Fermionic Characters and Arbitrary Highest-Weight Integrable $\hat{sl}_{r+1}$-Modules

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Abstract: This paper contains the generalization of the Feigin-Stoyanovskii construction to all integrable $\hat{sl}_{r+1}$-modules. We give formulas for the $q$-characters of any highest-weight integrable module of $\hat{sl}_{r+1}$ as a linear combination of the fermionic $q$-characters of the fusion products of a special set of integrable modules. The coefficients in the sum are the entries of the inverse matrix of generalized Kostka polynomials in $q^{-1}$. We prove the conjecture of Feigin and Loktev regarding the $q$-multiplicities of irreducible modules in the graded tensor product of rectangular highest weight-modules in the case of $\hat{sl}_{r+1}$. We also give the fermionic formulas for the $q$-characters of the (non-level-restricted) fusion products of rectangular highest-weight integrable $\hat{sl}_{r+1}$-modules.

1. Introduction

Fermionic formulæ for characters of highest-weight modules of affine algebras or vertex algebras first appeared in a purely algebraic context [17]. They were later shown [13, 12] to be related to the partition functions of certain statistical mechanical systems at their critical points. These character formulæ have desirable combinatorial properties, such as the manifest positivity of the coefficients that represent weight-space multiplicities. They also have a physical significance because they reflect the quasi-particle content of the statistical mechanical system. Consequently, algebraic constructions of bases for representations which reveal this combinatorial structure are important, and have been studied using several methods in the past dozen years.

One such method is that of Feigin and Stoyanovskii [23]. These authors used a theorem of Primc [21] to give an interesting construction of the vacuum integrable modules of the affine algebra $\hat{\mathfrak{g}}$ associated to any simple Lie algebra $\mathfrak{g}$. Their construction relies on the loop generators of the affine algebra. Physical systems associated with such integrable $\hat{\mathfrak{g}}$-modules are generalizations of the Heisenberg spin chain in statistical mechanics, or the WZW model in conformal field theory.
The formulæ of Feigin-Stoyanovskii [23] have an attractive interpretation in terms of (a bosonic version of) non-abelian quantum Hall states [19, 2]. In these states there are \( r \) “types” of particles that obey a generalized exclusion principle: the wave function vanishes if any \( k+1 \) particles occupy the same state. Here \( r \) is the rank of the algebra and \( k \) is the level of the integrable \( \hat{g} \)-module. In the presence of quasi-particle excitations, the wave functions can also vanish if fewer than \( k+1 \) particles occupy the same state. The statistics of the quasi-particles is ‘dual’ to the statistics of the fundamental particles [1].

The original construction of Feigin-Stoyanovskii can be used to compute [23] characters of vacuum (with highest weight \( k\Lambda_0 \)) representations of affine algebras. Later, Georgiev [10, 9] generalized it to some modules in the ADE series, with particularly simple highest weights, of the form \( l\omega_j + k\Lambda_0 \), corresponding to special rectangular Young diagrams. (Here \( \omega_j \) are certain fundamental \( g \)-weights, and \( l \in \mathbb{Z}_{\geq 0} \).)

In general, no fermionic formulæ are available for arbitrary highest-weight, integrable \( \hat{g} \)-modules. In this paper, we resolve this problem for the case of \( \hat{sl}_{r+1} \).

We explain, in terms of the functional realization of Feigin and Stoyanovskii, why such ‘rectangular highest weight’ modules are very special, and why there is no direct fermionic construction for other modules. However, we prove that it is possible to compute the character of any module as a finite sum of fermionic characters of the ‘rectangular’ highest-weight modules. The coefficients in this sum are the entries of the inverse matrix of generalized Kostka polynomials. These coefficients are, however, not manifestly positive (or even of positive degree).

In our construction we are naturally led to the graded tensor product of Feigin and Loktev [8] of finite-dimensional \( g \)-modules. In the case of irreducible \( sl_{r+1} \)-modules with highest weights of the form \( l\omega_j \) (where \( \omega_j \) is any fundamental weight), we compute the explicit fermionic form of the graded multiplicities of irreducible modules in the Feigin-Loktev tensor product, thus proving two of the conjectures of [8]: That the graded tensor product in this case is independent of the evaluation parameters, and that it is related to the generalized Kostka polynomials of [22, 16].

The plan of the paper is as follows. In Sect. 2 we give the basic definitions of the algebra and its modules. In Sects. 3 and 4, we supply the details of the generalized construction of [23] for integrable modules of \( \hat{sl}_{r+1} \), with highest weights corresponding to rectangular Young diagrams. In Sect. 5, we explain a similar calculation of graded characters of conformal blocks or coinvariants (the fusion product of [8]), which turn out to be related to the generalized Kostka polynomials of [22, 16]. We then use this calculation in Sect. 6 to compute the characters of arbitrary highest-weight representations. See Theorem 6.3 for the main result.

Although, for the sake of clarity, we concentrate in this paper on the case of \( \hat{g} = \hat{sl}_{r+1} \), the generalization to affine algebras associated with other simple Lie algebras is possible, but in that case one should replace the notion of integrable \( \hat{g} \)-modules with irreducible \( g \)-modules as their top component with those which have (the degeneration to the classical case of) Kirillov-Reshetikhin modules as their top component. We will give this construction in a future publication.

2. Notation

2.1. Current generators of affine algebras. Let \( g = sl_{r+1} \) and let \( \Pi = \{ \alpha_i \mid i = 1, \ldots, r \} \) denote its simple roots, and \( \{ \omega_i \mid i = 1, \ldots, r \} \) the fundamental weights. Let \( \{ e_{\alpha_i} = e_i \mid i = 1, \ldots, r \} \) denote the corresponding generators of \( n_+ \), and