A Generalization of Hawking’s Black Hole Topology Theorem to Higher Dimensions

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Received: 28 September 2005 / Accepted: 10 November 2005
Published online: 9 June 2006 – © Springer-Verlag 2006

Abstract: Hawking’s theorem on the topology of black holes asserts that cross sections of the event horizon in 4-dimensional asymptotically flat stationary black hole spacetimes obeying the dominant energy condition are topologically 2-spheres. This conclusion extends to outer apparent horizons in spacetimes that are not necessarily stationary. In this paper we obtain a natural generalization of Hawking’s results to higher dimensions by showing that cross sections of the event horizon (in the stationary case) and outer apparent horizons (in the general case) are of positive Yamabe type, i.e., admit metrics of positive scalar curvature. This implies many well-known restrictions on the topology, and is consistent with recent examples of five dimensional stationary black hole spacetimes with horizon topology $S^2 \times S^1$. The proof is inspired by previous work of Schoen and Yau on the existence of solutions to the Jang equation (but does not make direct use of that equation).

1. Introduction

A basic result in the theory of black holes is Hawking’s theorem [11, 13] on the topology of black holes, which asserts that cross sections of the event horizon in 4-dimensional asymptotically flat stationary black hole spacetimes obeying the dominant energy condition are spherical (i.e., topologically $S^2$). The proof is a beautiful variational argument, showing that if a cross section has genus $\geq 1$ then it can be deformed along a null hypersurface to an outer trapped surface outside of the event horizon, which is forbidden by standard results on black holes [13].¹ In [12], Hawking showed that his black hole topology result extends, by a similar argument, to outer apparent horizons in black hole spacetimes that are not necessarily stationary. (A related result had been shown by Gibbons [8] in the time-symmetric case.) Since Hawking’s arguments rely on the Gauss-Bonnet theorem, these results do not directly extend to higher dimensions.

¹ Actually the torus $T^2$ arises as a borderline case in Hawking’s argument, but can occur only under special circumstances.
Given the current interest in higher dimensional black holes, it is of interest to determine which properties of black holes in four spacetime dimensions extend to higher dimensions. In this note we obtain a natural generalization of Hawking’s theorem on the topology of black holes to higher dimensions. The conclusion in higher dimensions is not that the horizon topology is spherical; that would be too strong, as evidenced by the striking example of Emparan and Reall [7] of a stationary vacuum black hole spacetime in five dimensions with horizon topology $S^2 \times S^1$. The natural conclusion in higher dimensions is that cross sections of the event horizon (in the stationary case), and outer apparent horizons (in the general case) are of positive Yamabe type, i.e. admit metrics of positive scalar curvature. As noted in [6], in the time symmetric case this follows from the minimal surface methodology of Schoen and Yau [18] in their treatment of manifolds of positive scalar curvature. The main point of the present paper is to show that this conclusion remains valid without any condition on the extrinsic curvature of space. That such a result might be expected to hold is suggested by work in [19, Sect. 4], which implies that the apparent horizons corresponding to the blow-up of solutions of the Jang equation, as described in [19], are of positive Yamabe type. We emphasize, however, that we do not need to make use of the Jang equation here.2

Much is now known about the topological obstructions to the existence of metrics of positive scalar curvature in higher dimensions. While the first major result along these lines is the famous theorem of Lichnerowicz [16] concerning the vanishing of the $\hat{A}$ genus, a key advance in our understanding was made in the late 70’s and early 80’s by Schoen and Yau [17, 18], and Gromov and Lawson [9, 10]. A brief review of these results, relevant to the topology of black holes, was considered in [6]. We shall recall the situation in five spacetime dimensions in the next section, after the statement of our main result.

2. The Main Result

Let $V^n$ be an $n$-dimensional, $n \geq 3$, spacelike hypersurface in a spacetime $(M^{n+1}, g)$. Let $\Sigma^{n-1}$ be a closed hypersurface in $V^n$, and assume that $\Sigma^{n-1}$ separates $V^n$ into an “inside” and an “outside”. Let $N$ be the outward unit normal to $\Sigma^{n-1}$ in $V^n$, and let $U$ be the future directed unit normal to $V^n$ in $M^{n+1}$. Then $K = U + N$ is an outward null normal field to $\Sigma^{n-1}$, unique up to scaling.

The null second fundamental form of $\Sigma$ with respect to $K$ is defined as

$$\chi : T_p \Sigma \times T_p \Sigma \rightarrow \mathbb{R}, \quad \chi(X, Y) = \langle \nabla_X K, Y \rangle, \quad (2.1)$$

where $(\cdot, \cdot) = g$ and $\nabla$ is the Levi-Civita connection, of $M^{n+1}$. Then the null expansion of $\Sigma$ is defined as $\theta = \text{tr} \, \chi = h^{AB} \chi_{AB} = \text{div}_\Sigma K$, where $h$ is the induced metric on $\Sigma$.

We shall say $\Sigma^{n-1}$ is an outer apparent horizon in $V^n$ provided, (i) $\Sigma$ is marginally outer trapped, i.e., $\theta = 0$, and (ii) there are no outer trapped surfaces outside of $\Sigma$. The latter means that there is no $(n - 1)$-surface $\Sigma'$ contained in the region of $V^n$ outside of $\Sigma$ which is homologous to $\Sigma$ and which has negative expansion $\theta < 0$ with respect to its outer null normal (relative to $\Sigma$). Heuristically, $\Sigma$ is the “outer limit” of outer trapped surfaces in $V$.

2 In any case, the parametric estimates of [19] which are used to construct solutions of the Jang equation asymptotic to vertical cylinders over apparent horizons are generally true only in low dimensions.