Systems of Hess-Appel’rot Type

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Abstract: We construct higher-dimensional generalizations of the classical Hess-Appel’rot rigid body system. We give a Lax pair with a spectral parameter leading to an algebro-geometric integration of this new class of systems, which is closely related to the integration of the Lagrange bitop performed by us recently and uses Mumford relation for theta divisors of double unramified coverings. Based on the basic properties satisfied by such a class of systems related to bi-Poisson structure, quasi-homogeneity, and conditions on the Kowalevski exponents, we suggest an axiomatic approach leading to what we call the “class of systems of Hess-Appel’rot type”.

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1. Introduction. Starting from the Kowalevski Analysis

It is well known that Kowalevski, in her celebrated 1889 paper [28], starting with a careful analysis of the solutions of the Euler and the Lagrange case of rigid-body motion, formulated a problem of describing the parameters \((A, B, C, x_0, y_0, z_0)\), for which the Euler – Poisson equations have a general solution in the form of uniform functions only with moving poles as singularities. Here, \(I = \text{diag}(A, B, C)\) represents the inertia operator, and \(\chi = (x_0, y_0, z_0)\) is the centre of mass of the rigid body.

Then, in §1 of [28], some necessary conditions were formulated and a new case was discovered, now known as the Kowalevski case, as a unique case possible beside the cases of Euler and Lagrange. However, considering the situation where all momenta of inertia are different, Kowalevski came to a relation analogous to the following (see [24]):

\[
x_0 \sqrt{A(B - C)} + y_0 \sqrt{B(C - A)} + z_0 \sqrt{C(A - B)} = 0,
\]

and concluded that \(x_0 = y_0 = z_0\), giving the Euler case.

But, it was Appel’rot who noticed in the beginning of the 1890’s that the last relation admits one more case, not mentioned by Kowalevski:

\[
x_0 \sqrt{A(B - C)} + z_0 \sqrt{C(A - B)} = 0, \quad y_0 = 0,
\]

under the assumption \(A > B > C\). Such systems were considered also by Hess, even before Appel’rot, in 1890. Such an intriguing position corresponding to the one overlooked in the Kowalevski paper, made the Hess-Appel’rot systems very attractive for leading Russian mathematicians from the end of the XIX century. After a few years, Nekrasov and Lyapunov managed to provide new arguments and they demonstrated that the Hess-Appel’rot systems didn’t satisfy the condition investigated by Kowalevski, which means that the conclusion of §1 of [28] was correct.

And, from that moment, the Hess-Appel’rot systems were basically left aside, even in modern times, when new methods of inverse problems, Lax representations, finite-zone integrations were applied to almost all known classical systems, until very recently.

A few years ago, we constructed a Lax representation for the Hess-Appel’rot system (see [15]).

Now, in this paper the first higher - dimensional generalizations of the Hess-Appel’rot systems are constructed. For each dimension \(n > 3\), we give a family of such generalizations. We provide Lax representations for all new systems, generalizing the Lax pair from [15]. We show that the new systems are isoholomorphic. This class of systems was introduced and studied in [16], in connection with the Lagrange bitop.

Lax matrices of isoholomorphic systems have specific distributions of zero entries. Therefore standard integration techniques of [17, 1] cannot be applied directly. Its integration requires more detailed analysis of geometry of Prym varieties and it is based on Mumford’s relation on theta - divisors of unramified double coverings.

In the present paper, in addition, we perform in detail the integration procedure in the first higher-dimensional case \(n = 4\) of new Hess-Appel’rot type systems.

The \(L\)-operator, a quadratic polynomial in \(\lambda\) of the form \(\lambda^2 C + \lambda M + \Gamma\), in the case \(n = 4\), satisfies the condition

\[
L_{12} = L_{21} = L_{34} = L_{43} = 0.
\]

Such situation, explicitly excluded by Adler-van Moerbeke (see [1], Theorem 1) and implicitly by Dubrovin (see [17], Lemma 5 and Corollary) was studied for the first time.