Quantum Liouville Theory in the Background Field Formalism I. Compact Riemann Surfaces

Leon A. Takhtajan\textsuperscript{1}, Lee-Peng Teo\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651, USA. E-mail: leontak@math.sunysb.edu

\textsuperscript{2} Faculty of Information Technology, Multimedia University, Jalan Multimedia, Cyberjaya 63100, Selangor, Malaysia. E-mail: lpteo@mmu.edu.my

Received: 13 September 2005 / Accepted: 29 March 2006
Published online: 23 August 2006 – © Springer-Verlag 2006

Abstract: Using Polyakov’s functional integral approach and the Liouville action functional defined in [ZT87c] and [TT03a], we formulate quantum Liouville theory on a compact Riemann surface $X$ of genus $g > 1$. For the partition function $\langle X \rangle$ and correlation functions with the stress-energy tensor components $\langle \prod_{i=1}^{n} T(z_i) \prod_{k=1}^{l} \bar{T}(\bar{w}_k) X \rangle$, we describe Feynman rules in the background field formalism by expanding corresponding functional integrals around a classical solution, the hyperbolic metric on $X$. Extending analysis in [Tak93, Tak94, Tak96a, Tak96b], we define the regularization scheme for any choice of the global coordinate on $X$. For the Schottky and quasi-Fuchsian global coordinates, we rigorously prove that one- and two-point correlation functions satisfy conformal Ward identities in all orders of the perturbation theory. Obtained results are interpreted in terms of complex geometry of the projective line bundle $E_c = \lambda_c H^{c/2}$ over the moduli space $\mathcal{M}_g$, where $c$ is the central charge and $\lambda_H$ is the Hodge line bundle, and provide the Friedan-Shenker [FS87] complex geometry approach to CFT with the first non-trivial example besides rational models.

Contents

1. Introduction ................................... 136
2. Classical Liouville Theory ........................... 142
   2.1 Liouville action functional ................................ 143
   2.2 The stress-energy tensor ................................ 145
3. Quantum Liouville Theory ........................... 147
   3.1 Feynman rules for the partition function ....................... 147
   3.2 Feynman rules for correlation functions ....................... 151
4. Deformation Theory ............................... 154
   4.1 Schottky and Teichmüller spaces ........................... 154
   4.2 Formal geometry on deformation spaces ...................... 156
   4.3 Variational formulas ............................... 156
1. Introduction

Classical Liouville theory is a Euclidean field theory associated with hyperbolic Riemann surfaces. Complete conformal metrics $ds^2$ on a Riemann surface $X$ are classical fields of the theory, and the so-called Liouville equation—the equation $K(ds^2) = -1$, where $K(ds^2)$ is a Gaussian curvature, is the corresponding Euler-Lagrange equation. According to the uniformization theorem, it has a unique solution—the hyperbolic metric on $X$. The quantized Liouville theory describes “quantum corrections” to hyperbolic geometry of $X$ by taking into account fluctuations around the hyperbolic metric. In 1981, Polyakov formulated a functional integral approach to bosonic string theory, and made a fundamental discovery that quantum Liouville theory is a conformal anomaly for non-critical strings [Pol81]. Thus in order to find correlation functions of vertex operators of the bosonic string in any dimension $D$ (and not only for $D = 26$), one needs to know correlation functions of the Liouville vertex operators $V_\alpha(z) = e^{\alpha \phi(z)}$, where $ds^2 = e^{\phi(z)}|dz|^2$ is the Liouville field—a conformal metric on $X$. The fundamental property that classical fields and equation of motion of the Liouville theory are conformally invariant, led Belavin, Polyakov and Zamolodchikov to their formulation of the two-dimensional Conformal Field Theory (CFT) [BPZ84]. Though the problem of computing correlation functions of the Liouville vertex operators, needed for non-critical string theory, is still outstanding, in the works of Dorn and Otto [DO94], and of Zamolodchikov and Zamolodchikov [ZZ96] the quantum Liouville theory was formulated as a non-rational model of CFT with a continuous spectrum of conformal dimensions (see the review [Tes01] for a complete account and references).

In [Pol82], Polyakov proposed a functional integral representation for correlation functions of the Liouville vertex operators in the form needed for the non-critical string theory. This so-called geometric approach to the quantum Liouville theory was formalized and developed in [Tak93, Tak94, Tak96a, Tak96b]. In this formulation, correlation functions of Liouville vertex operators on the Riemann sphere $\mathbb{P}^1$ are defined by