Energy Splitting, Substantial Inequality, and Minimization for the Faddeev and Skyrme Models

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Abstract: In this paper, we prove that the Faddeev energy $E_1$ at the unit Hopf charge is attainable. The proof is based on utilizing an important inequality called the substantial inequality in our previous paper which describes how the Faddeev energy splits into its sublevels in terms of energy and topology when compactness fails. With the help of an optimal Sobolev estimate of the Faddeev energy lower bound and an upper bound of $E_1$, we show that $E_1$ is attainable. For the two-dimensional Skyrme model, we prove that the substantial inequality is also valid, which allows us to greatly improve the range of the coupling parameters for the existence of unit-charge solitons previously guaranteed in a smaller range of the coupling parameters by the validity of the concentration-compactness method.

1. Introduction

Global energy minimizers are important in field theory as they provide leading-order contributions to the transition amplitudes calculated through functional integrals or partition functionals for the quantization of fundamental particle systems [9]. Some prototype examples include kinks, vortices, monopoles, and instantons, which are static solitons characterized by various topological invariants. Except for the one-dimensional (1D) kink case which is completely integrable, in all the other cases, global energy minimizers can only be obtained in the so-called BPS limits. The main difficulty we encounter in this kind of problems is a lack of compactness because the energy functionals are all defined over the full Euclidean spaces. For the well-known Skyrme model and the Faddeev model, the situation is even less transparent because these models do not have a BPS-limit structure. Therefore, one is forced to study the direct minimization problem for these models. From an analytic point of view, the first temptation would be to try to see whether the concentration-compactness method [14] works because this method is developed to tackle similar minimization problems defined over full spaces which says that a minimizing sequence converges (hence compactness holds) if after suitable
translations it concentrates in a local region (that is, if concentration takes place). For our problems, however, it is not directly possible to establish such a concentration-compactness picture. In fact, we will have to be forced to study the situation when concentration-compactness fails and an energy splitting or dichotomy takes place. It is interesting that the topological structure of these problems now become important which allows us to deduce concentration-compactness indirectly from an inequality we call “the substantial inequality” which originates essentially from assuming dichotomy or energy splitting. We have seen in [12] that this substantial inequality method enabled us to establish a series of existence theorems for the Faddeev model [6–8] and the 3D Skyrme model [18–21, 27], which were previously unavailable. In this paper, we will use this method to establish the much anticipated existence theorem that the Faddeev energy \( E_1 \) at the unit Hopf charge is attainable. Besides, we will use the same method to establish some new existence results for the 2D Skyrme model which considerably improve the existence result previously obtained in [13] using the concentration-compactness method.

The rest of this paper is organized as follows. In the next section, we recall the existence problem of the Faddeev model and prove that the Faddeev energy \( E_1 \) at the unit Hopf charge is attainable by using the substantial inequality method. This method relies on some suitable energy estimates which are consequences of a specific topological energy lower bound and an upper estimate for \( E_1 \), which will be elaborated in detail in Sect. 3 and Sect. 4. In Sect. 5, we study the 2D Skyrme model and we prove that the substantial inequality is valid. In particular, we show that the minimization problem of the 2D Skyrme model has a solution within a suitable (but unknown) topological class. In Sect. 6, we use the substantial inequality method as we do for the Faddeev model to show the existence of a least-positive-energy minimizer for the 2D Skyrme model. We also show that an energy minimizer for the 2D Skyrme model exists at the unit topological degree when the product of the coupling constants lies in an explicit interval which greatly improves the interval we obtained in [13] by using the concentration-compactness method directly. We also remark that the values of the coupling constants in the Faddeev model and Skyrme model are not important for the understanding of their minimization problems.

2. Minimization for the Faddeev Model

Let \( \mathbf{n} = (n_1, n_2, n_3) : \mathbb{R}^3 \to S^2 \) be a map (from the Euclidean 3-space to the unit 2-sphere) and \( F_{jk}(\mathbf{n}) = \mathbf{n} \cdot (\partial_j \mathbf{n} \wedge \partial_k \mathbf{n}) \) \((j, k = 1, 2, 3)\) the induced (Faddeev) magnetic field. We follow [25] to use the renormalized Faddeev energy

\[
E(\mathbf{n}) = \int_{\mathbb{R}^3} \left\{ \sum_{1 \leq k \leq 3} |\partial_k \mathbf{n}|^2 + \frac{1}{2} \sum_{1 \leq k < \ell \leq 3} F^2_{kk}(\mathbf{n}) \right\} \, dx
= \int_{\mathbb{R}^3} \left( |\nabla \mathbf{n}|^2 + \frac{1}{2} |\mathbf{F}|^2 \right) \, dx. \tag{2.1}
\]

Here \( \mathbf{F} = \mathbf{F}(\mathbf{n}) = (\frac{1}{2} \epsilon^{jkk'} F_{kk'}(\mathbf{n})) = (F_{23}(\mathbf{n}), -F_{13}(\mathbf{n}), F_{12}(\mathbf{n})) \). The finite-energy condition implies that \( \mathbf{n} \) approaches a constant vector \( \mathbf{n}_\infty \) at infinity of \( \mathbb{R}^3 \). Hence we may compactify \( \mathbb{R}^3 \) into \( S^3 \) and view the fields as maps from \( S^3 \) to \( S^2 \). As a consequence, we see that each finite-energy field configuration \( \mathbf{n} \) is associated with an integer, \( Q(\mathbf{n}) \), in \( \pi_3(S^2) = \mathbb{Z} \). In fact, such an integer \( Q(\mathbf{n}) \) is known as the Hopf invariant which has