Nonlinear Instability for the Navier-Stokes Equations

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Abstract: It is proved, using a bootstrap argument, that linear instability implies nonlinear instability for the incompressible Navier-Stokes equations in $L^p$ for all $p \in (1, \infty)$ and any finite or infinite domain in any dimension $n$.

1. Introduction

The stability/instability of a flow of viscous incompressible fluid governed by the Navier-Stokes equations is a classical subject with a very extensive literature over more than 100 years. Much of the classical literature has concerned the stability of relatively simple specific flows (e.g. Couette flows and Poiseuille flows), the spectrum of the Navier-Stokes equations linearized about such flows and the role of the critical Reynolds number delineating the linearly stable and unstable regimes. An elegant result for general bounded flows was proved by Serrin [18] who used energy methods to show that all flows are nonlinearly stable in $L^2$ norm when the Reynolds number is less than a specific constant ($\pi \sqrt{3}$). Hence all steady flows that are sufficiently slow or sufficiently viscous are stable. However for many physical situations the Reynolds number is much larger than $\pi \sqrt{3}$, often by many orders of magnitude and observations indicate that such flows are unstable.

Linear instability has been confirmed in some specific examples by demonstrating existence of a nonempty unstable spectrum for the linearized Navier-Stokes operator. For example, Meshalkin and Sinai [14] used Fourier series and continued fractions to show the existence of unstable eigenvalues in the case of so-called Kolmogorov flows (i.e. plane parallel shear flow with a sinusoidal profile). Sattinger [16] uses a Galerkin argument to prove that linear instability implies nonlinear instability for weak solutions in $L^2$ in the case of bounded domains. In a book published in Russian in 1984 (and in English in 1989) Yudovich [21] obtained an important result relating linear stability/instability for the Navier-Stokes equations with nonlinear stability/instability (see also Henry [8]). These results were proved in the function space $L^q(\Omega)$ with $q \geq n$ in $n$-spatial dimensions. A fairly general abstract theorem of Friedlander et al. [5] can be
applied to the Navier-Stokes equations in a finite domain to prove nonlinear instability in $H^s$, $s > \frac{n}{2} + 1$ when the linearized operator has an unstable eigenvalue in $L^2$.

In this present paper we extend the result that linear instability implies nonlinear instability for the Navier-Stokes equations to all $L^p$ spaces with $1 < p < \infty$ and both finite domains and $\mathbb{R}^n$. We note that our result includes nonlinear instability in the $L^2$ energy norm which we claim is the natural norm in which to consider issues of stability and instability.

The technique we employ to prove our main result is a bootstrap argument. Such arguments have been previously employed by several authors to prove under certain restrictions that linear instability implies nonlinear instability for the 2 dimensional Euler equation (Bardos et al. [1], Friedlander and Vishik [20], Lin [13]). Because in general the spectrum of the Euler operator has a continuous component, unlike the Navier-Stokes operator in a finite domain whose spectrum is purely discrete, these nonlinear instability results for the Euler equation are much more limited than those presented here for the Navier-Stokes equations.

2. Notation and Formulation

We consider solutions to the Navier-Stokes equations

$$\frac{\partial q}{\partial t} = -(q \cdot \nabla)q - \nabla p + R^{-1} \Delta q + f,$$

(2.1a)

$$\nabla \cdot q = 0,$$

(2.1b)

where $q(x,t)$ denotes the $n$-dimensional velocity vector, $p(x,t)$ denotes the pressure and $f(x)$ is an external force vector. The dimensionless parameter $R$ is the Reynolds number defined as $R = \frac{V L}{\nu}$, where $V$ and $L$ are characteristic velocity and length scales of the system and $\nu$ is the viscosity of the fluid. In Sect. 3 we consider the system on the $n$-dimensional torus $\mathbb{T}^n$ and in a bounded domain $\Omega \subset \mathbb{R}^n$. In Sect.4 we consider the system in $\mathbb{R}^n$. The results are valid in all dimensions $n$ although the most relevant physical cases are $n = 2$ and 3. We impose the standard boundary conditions on solutions of (2.1) for each type of domain: the no-slip condition $q|_{\partial \Omega} = 0$, in the case of $\Omega$; vanishing velocity $q(x) \to 0$, as $x \to \infty$, in the case of $\mathbb{R}^n$; and periodic boundary condition in case of the torus. The results in Sects.3 and 4 prove that spectral instability for the linearized Navier-Stokes equations implies nonlinear instability in $L^p$ for $1 < p < \infty$. In Sect.5 we prove a result relating spectral stability with nonlinear stability in $L^p$ for $p > n$.

Here and thereafter, for any $p \in [1, \infty)$, $L^p$ denotes the usual Lebesgue space, with norm denoted $\| \cdot \|_p$, intersected with the space of divergence free functions. We let $W^{s,p}$ stand for the Sobolev space in the same context with norm denoted $\| \cdot \|_{s,p}$.

We consider an arbitrary steady solution of (2.1),

$$0 = -(U_0 \cdot \nabla)U_0 - \nabla P_0 + R^{-1} \Delta U_0 + f,$$

(2.2a)

$$\nabla \cdot U_0 = 0.$$  

(2.2b)

We assume $U_0(x) \in C^\infty$ and $f(x) \in C^\infty$. To discuss stability of $U_0$ we rewrite the Navier-Stokes equations (2.1) in perturbation form with $q(x,t) = U_0(x) + v(x,t)$,

$$\frac{\partial v}{\partial t} = -(U_0 \cdot \nabla)v - (v \cdot \nabla)U_0 + R^{-1} \Delta v - \nabla \cdot (v \otimes v) - \nabla p,$$

(2.3a)

$$\nabla \cdot v = 0,$$

(2.3b)

$$v|_{t=0} = v_0.$$  

(2.3c)