The Expected Area of the Filled Planar Brownian Loop is $\pi/5$

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Abstract: Let $B_t$, $0 \leq t \leq 1$ be a planar Brownian loop (a Brownian motion conditioned so that $B_0 = B_1$). We consider the compact hull obtained by filling in all the holes, i.e. the complement of the unique unbounded component of $\mathbb{C} \setminus B[0, 1]$. We show that the expected area of this hull is $\pi/5$. The proof uses, perhaps not surprisingly, the Schramm Loewner Evolution (SLE). As a consequence of this result, using Yor's formula [17] for the law of the index of a Brownian loop, we find that the expected area of the region inside the loop having index zero is $\pi/30$; this value could not be obtained directly using Yor's index description.

1. Introduction

The main result of the present paper goes as follows: Let $B$ denote a Brownian loop in $\mathbb{C}$ of time duration 1. There are various equivalent ways to define it. One can view it as a Brownian path $(B_t, 0 \leq t \leq 1)$ appropriately conditioned to be back at its starting point at time 1. One can also write $B_t = W_t - t W_1$, where $W$ is just a standard Brownian motion in $\mathbb{C}$. Then, $\mathbb{C} \setminus B[0, 1]$, i.e. the complement of the path, has a unique infinite connected component $H$. The hull $T$ generated by the Brownian loop is by definition equal to $\mathbb{C} \setminus H$. This is the set obtained by filling in the holes in the loop. Let $A$ be the random variable whose value is the area of $T$. Then:

**Theorem 1.1.** The expected value of $A$ is $\pi/5$.

Our result gives interesting information regarding the Brownian loop soups introduced in [4]. This conformally invariant object plays an important role in the understanding and description of SLE curves (see, e.g. [4, 14, 5]). It can be viewed as a Poissonian cloud (of intensity $c$) of filled Brownian loops in subdomains of the plane. Among other things, it is announced in [15] that the dimension of the set of points in the complement of the loop soup (i.e. the points that are in the inside of no loop) can be shown to be equal to $2 - c/5$, using consequences of the restriction property. A detailed proof of this
The statement has never been published, and in fact, our result implies the corresponding first moment estimate (i.e., the mean number of balls of radius \( \varepsilon \) needed to cover the set). The other arguments needed to derive the result announced in [15] will be detailed in [9].

Another consequence of our result concerns the direct relation between different measures on self-avoiding loops in the plane defined as outer boundaries of planar Brownian loops. See [16].

In the abundant existing literature about planar Brownian motion, there are certainly results dealing with the question of area. Paul Lévy’s stochastic area formula describing the algebraic area “swept” by a Brownian motion will likely come to the mind of many readers. Our result, however, is very different from this classical theorem, firstly because Lévy’s area is a signed area, but mainly because of the following: in order to apprehend Lévy’s area it is enough to follow the Brownian curve locally without paying attention to the rest of the curve. In our case, one needs to consider the curve globally.

Also, Yor [17] has been able to give an explicit formula for the law of the index of a Brownian loop around a fixed point \( z \). Yor’s proof relies on the fact that the index can be obtained via a stochastic integral along the loop. Let us explain how this result is related to ours. A point with a non-zero index has to be inside the loop. Using this fact, it is almost possible to describe the probability that a given point is inside the loop, modulo the problem of the zero index; indeed, there are some regions inside the Brownian loop around which the loop has an index equal to zero. In the last section of our paper, we combine Theorem 1.1 with the law of the index given by Yor, to find that the expected area of the regions of index \( n \in \mathbb{Z} \setminus \{0\} \) can be directly obtained by integrating Yor’s formula with respect to \( z \).

In [2], using physics methods, Comtet, Desbois and Ouvry obtained the values of the expected areas for the non-zero index regions by different techniques, and they pointed out the different nature of the \( n = 0 \) sector (the points in the plane of zero index) and emphasized that “it would be interesting to distinguish in the \( n = 0 \) sector, curves which do not enclose the origin from curves which do enclose the origin but an equal number...