Pseudodifferential Symbols on Riemann Surfaces and Krichever–Novikov Algebras

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Abstract: We define the Krichever-Novikov-type Lie algebras of differential operators and pseudodifferential symbols on Riemann surfaces, along with their outer derivations and central extensions. We show that the corresponding algebras of meromorphic operators and symbols have many invariant traces and central extensions, given by the logarithms of meromorphic vector fields. Very few of these extensions survive after passing to the algebras of operators and symbols holomorphic away from several fixed points. We also describe the associated Manin triples and KdV-type hierarchies, emphasizing the similarities and differences with the case of smooth symbols on the circle.

1. Introduction

The Krichever-Novikov algebras are the (centrally extended) Lie algebras of meromorphic vector fields on a Riemann surface $\Sigma$, which are holomorphic away from several fixed points [7, 8], see also [11, 15]. They are natural generalizations of the Virasoro algebra, which corresponds to the case of $\Sigma = \mathbb{C}P^1$ with two punctures. Central extensions of the corresponding algebras of vector fields on a given Riemann surface are defined by fixing a projective structure (that is a class of coordinates related by projective transformations) and the corresponding Gelfand-Fuchs cocycle, along with the change-of-coordinate rule.

In this paper we deal with two generalizations of the Krichever-Novikov (KN) algebras. The first one is the Lie algebras of all meromorphic differential operators and pseudodifferential symbols on a Riemann surface, while the second one is the Lie algebras of meromorphic differential operators and pseudodifferential symbols which are holomorphic away from several fixed points. The main tool which we employ is fixing a reference meromorphic vector field instead of a projective structure on $\Sigma$. It turns out that such a choice allows one to write more explicit formulas for the corresponding cocycles, both for the Krichever-Novikov algebra of vector fields and for its generalizations.
Several features of these algebras of meromorphic symbols make them different from their smooth analogue, the algebra of pseudodifferential symbols with smooth coefficients on the circle. First of all, this is the existence of many invariant traces on the former algebras: one can associate such a trace to every point on the surface. Furthermore, we show that the logarithm \( \log X \) of any meromorphic pseudodifferential symbol \( X \) defines an outer derivation of the Lie algebra of meromorphic symbols. In turn, the combination of invariant traces and outer derivations produces a variety of independent non-trivial 2-cocycles on the Lie algebras of meromorphic pseudodifferential symbols and differential operators, as well as it gives rise to Lie bialgebra structures (see Sect. 2). Note that the above mentioned scheme of generating numerous 2-cocycles in the meromorphic case, which involve \( \log X \) for any meromorphic pseudodifferential symbol \( X \), provides a natural unifying framework for the existence of two independent cocycles (generated by \( \log \partial/\partial x \) and \( \log x \)) in the smooth case, cf. [6, 5].

The second type of algebras under consideration, those of holomorphic differential operators and pseudodifferential symbols, are more direct generalizations of the Krich-ever-Novikov algebra of holomorphic vector fields on a punctured Riemann surface. For them we prove the density and filtered generalized grading properties, similarly to the corresponding properties of the KN algebras [7, 8]. Furthermore, one can adapt the notion of a local cocycle proposed in [7] to the filtered algebras of (pseudo)differential symbols. It turns out that all logarithmic cocycles become linearly dependent when we confine to local cocycles on holomorphic differential operators. On the other hand, for holomorphic pseudodifferential symbols the local cocycles are shown to form a two-dimensional space (see Sect. 3).

Finally, for meromorphic differential operators, as well as for holomorphic differential operators on surfaces with trivialized tangent bundle, there exist Lie bialgebra structures and integrable hierarchies mimicking the structures in the smooth case.

We deliberately put the exposition in a form which emphasizes the similarities with and differences from the algebras of (pseudo)differential symbols with smooth coefficients on the circle, developed in [3, 5]. In many respects the algebras of holomorphic symbols extended by local 2-cocycles turn out to be similar to their smooth counterparts on the circle. On the other hand, by giving up the condition of locality, one obtains higher-dimensional extensions of the Lie algebras of holomorphic symbols by means of the 2-cocycles related to different paths on the surface. This way one naturally comes to holomorphic analogues of the algebras of “smooth symbols on graphs,” which also have central extensions given by 2-cocycles on different loops in the graphs.

2. Meromorphic Pseudodifferential Symbols on Riemann Surfaces

2.1. The algebras of meromorphic differential and pseudodifferential symbols. Let \( \Sigma \) be a compact Riemann surface and \( \mathcal{M} \) be the space of meromorphic functions on \( \Sigma \). Fix a meromorphic vector field \( v \) on the surface and denote by \( D \) (or \( D_v \)) the operator of Lie derivative \( L_v \) along the field \( v \). Then \( D \) sends the space \( \mathcal{M} \) to itself, and one can consider the operator algebras generated by it.

**Definition 2.1.** The associative algebras of meromorphic differential operators

\[
MDO := \left\{ A = \sum_{k=0}^{n} a_k D^k \mid a_k \in \mathcal{M} \right\}
\]

