Discrete Path Integral Approach to the Selberg Trace Formula for Regular Graphs *

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Abstract: We give a new proof of the Selberg trace formula for regular graphs. Our approach is inspired by path integral formulation of quantum mechanics, and calculations are mostly combinatorial.

1. Introduction

The famous Selberg trace formula first appeared in [1]. On a compact hyperbolic surface it relates the eigenvalue spectrum of the Laplace operator to the length spectrum of closed geodesics. A version of this formula for finite regular graphs was obtained by Ahumada [2] (cf. also Ihara [3]).

Trace formulae are known to have many implications. For instance, they can be considered as nonabelian generalizations of the Poisson summation formula. In case of finite graphs, since one can find the eigenvalue spectrum of the Laplacian for a given graph explicitly, the trace formula lets one find the numbers of closed geodesics of any length (see (34)). In physics trace formulae indicate the cases when semi-classical evaluation of the path integral for state sum of a quantum free particle in some background is exact. The idea of deriving the original Selberg trace formula for hyperbolic surfaces from a path integral belongs to Gutzwiller [6]. The Selberg trace formula is also known to bear much resemblance to the Riemann-Weil formula in number theory.

The original proof of the trace formula for regular graphs (30) was in the framework of “discrete harmonic analysis”. We propose another way to derive it, inspired by the path integral approach in quantum mechanics [5]. We consider the trace $Z_\Delta(t) = \text{tr} e^{t\Delta}$ as a state sum of the quantum free particle living on the graph. We rewrite it as a sum over closed paths, which is a discrete version of the usual path integral over loops for the quantum mechanical state sum. Then we divide the set of closed paths into classes of homotopically equivalent paths. There is a class of contractible paths, and one homotopy

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class for each closed geodesic on the graph. We explicitly calculate the contribution of each homotopy class to the state sum, thus rewriting it as a contribution of contractible paths plus the sum over “nontrivial” geodesics (of nonzero length) of contributions of their individual homotopy classes. This is analogous to the stationary phase calculation of the path integral. Geodesics serve as stationary points of the action in the space of loops. The homotopy class of a geodesic serves as a neighbourhood of the stationary point. Thus we arrive at the known trace formula for a regular graph, with a specific, physically relevant, choice of test function for the eigenvalue spectrum of the Laplacian.

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2. Notations and Definitions

Let $\Gamma$ be a finite regular connected non-oriented graph with vertices of valence $q + 1 \geq 2$ with no multiple edges and no edges connecting a vertex with itself. Denote by $V(\Gamma)$ and $E(\Gamma)$ the set of vertices of $\Gamma$ and the set of edges respectively. Let $|\Gamma| = |V(\Gamma)|$ be the number of vertices. Further denote the space of complex-valued functions on vertices by $\text{Fun}(\Gamma) = \mathbb{C}^{V(\Gamma)}$. A basis function (vector) $|v\rangle$ associated with vertex $v$ equals 1 on $v$ and 0 on the other vertices. We further adopt the quantum-mechanical notations and denote the transposed basis vector by $<v| = |v\rangle^T$. We call the set of vertices connected to $v$ by edges its link and denote it $\text{Lk}(v)$.

The averaging operator $T : \text{Fun}(\Gamma) \to \text{Fun}(\Gamma)$ acts as follows: for $f \in \text{Fun}(\Gamma)$,

$$ (Tf)(v) = \sum_{v' \in \text{Lk}(v)} f(v'). \quad (1) $$

The Laplace operator $\Delta$ on $\Gamma$ is defined by

$$ (\Delta f)(v) = \sum_{v' \in \text{Lk}(v)} f(v') - \text{val}(v) f(v), \quad (2) $$

where $\text{val}(v)$ is the valence of $v$. Since we consider a regular graph $\Gamma$, $\Delta$ differs from $T$ by a multiple of identity: $\Delta = -(q + 1)I + T$, where $I$ is the identity map $\text{Fun}(T) \to \text{Fun}(T)$. The matrix of the averaging operator is just the adjacency matrix of the graph: $<v'|T|v> = 1$ if $v$ and $v'$ are connected by an edge and 0 otherwise. The diagonal elements of $T$ are zero.

The physically interesting quantity is the trace of the heat kernel (the state sum) $e^{t\Delta}$,

$$ Z_\Delta(t) = \text{tr} \exp(t\Delta) = e^{-(q+1)t} Z_T(t), \quad (3) $$

where

$$ Z_T(t) = \text{tr} \exp(tT). \quad (4) $$

It turns out that $Z_T(t)$ is more convenient for our calculation than $Z_\Delta(t)$. If $\{\lambda_j\}_{j=1}^{\Gamma}$ is the set of eigenvalues of $T$ then

$$ Z_T(t) = \sum_{j=1}^{|\Gamma|} e^{\lambda_j t}. \quad (5) $$