Asymptotics of Block Toeplitz Determinants and the Classical Dimer Model

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Abstract: We compute the asymptotics of a block Toeplitz determinant which arises in the classical dimer model for the triangular lattice when considering the monomer-monomer correlation function. The model depends on a parameter interpolating between the square lattice \((t = 0)\) and the triangular lattice \((t = 1)\), and we obtain the asymptotics for \(0 < t \leq 1\). For \(0 < t < 1\) we apply the Szegő Limit Theorem for block Toeplitz determinants. The main difficulty is to evaluate the constant term in the asymptotics, which is generally given only in a rather abstract form.

1. Introduction

A dimer is a bond connecting two nearest neighbors on a lattice. There is a long history in statistical mechanics of the study of the configurations of the bonds on planar lattices and mathematical results involving determinants. The most classical connection, due to Fisher and Stephenson [9], is the reformulation of the two-dimensional Ising model as a dimer model on a decorated planar lattice. Since the dimer model can be reduced, by means of the theorem of Kasteleyn [12, 14], to the computation of a certain Pfaffian, the problem is then equivalent to computing a determinant.

Recently, there has been increased interest in the dimer statistics due to the investigation of quantum dimer models. These models are important in the study high-temperature superconductivity. It has been proposed that the superconducting state evolves out of a short-range resonating valence bond (RVB) state. The underlying Hilbert space of these models consists of all configurations of pairings of the spins on the lattice into singlet bonds. In the work of Rokhsar and Kivelson on the square lattice, it was shown that this quantum dimer model exhibits a critical point in which the equal-amplitude superpositions of all dimer configurations of each topological sector are the unique zero-energy ground states. However, this “RK” point turned out to be an isolated point between two solid phases, rather than an actual RVB phase [13, 16, 17, 19].

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For the triangular lattice, the situation is quite different. In [15, 18] it was shown that there is an RVB phase characterized by liquid correlations. It turns out that ground state correlations at the “RK” point are the same as those computed from the correlations of the classical dimer model. Hence it is important to have information about such correlations. This was the motivation for the results obtained in this paper.

In the hard-core, close-packed dimer model, each site is paired with exactly one of its nearest neighbors. If the dimers have equal fugacity, each configuration is equally likely and the partition function is simply the number of configurations. For planar lattices this can be computed using classical Pfaffian techniques that go back to the work of Kasteleyn and Fisher [9, 12]. On the planar lattice a directed graph is constructed so that an arrow connects all nearest neighbors with the property that the number of clockwise arrows around any closed circuit is odd. Then a modified adjacency matrix $M_{i,j}$ is constructed so that

$$M_{i,j} = 1 \text{ if an arrow points from site } i \text{ to site } j, \quad M_{i,j} = -1 \text{ if the arrow points from site } j \text{ to } i \text{ and } 0 \text{ otherwise.}$$

For the triangular lattice with arrows pointing to the right/upwards, this approach is modified so that the same matrix is considered, except that in the lattice, alternating factors of $\pm i$ are placed on alternating rows. In either case the result is that the number of configurations is $(\det M)^{1/2}$. To interpolate, the triangular bonds are weighted by $t$ so that $t = 0$ corresponds to the square lattice and $t = 1$ corresponds to the triangular lattice case.

The monomer-monomer correlation is the ratio of square roots of determinants described above, but on different lattices. The denominator is for lattice points equally spaced. The numerator is the same, but for the lattice with two sites removed. For this paper, the sites are $n$ spaces apart in adjacent rows. If this ratio, denoted by $P^{(mm)}(n)$, tends to zero as $n$ tends to infinity, it is said that the model is confining. If it tends to a non-zero limit, it is said to be deconfining and is a desired feature for the 2d quantum models. In this paper we show that the limit is non-zero for $t > 0$. We also note that our computation yields a result for correlations with variable bonds, a result that we believe is only rarely explicitly obtainable.

The starting point for our work is the computation from [8], by Fendley, Moessner, and Sondhi which describes the monomer-monomer correlation function $P^{(mm)}(n)$, as the determinant of a block matrix

$$P^{(mm)}(n) = \frac{1}{2} \sqrt{\det M_n}, \quad M_n = \begin{pmatrix} \mathcal{R} & \mathcal{Q} \\ \mathcal{Q} & \mathcal{R} \end{pmatrix},$$

(1)

where $\mathcal{R}$ is an $n \times n$ matrix with entries depending on the difference of their indices, and $\mathcal{Q}$ is an $n \times n$ matrix with entries depending on the sum of the indices. The authors computed this correlation function numerically for $t = 1$ as the size of the matrix increases and found that it converged to a constant value of around 0.1494... For all the details of the dimer model computation, the reader is referred to [8].

We will not only compute the determinant asymptotically for all values of $t$ between zero and one but in addition we will be able to determine the asymptotics for all complex parameters $t$ with $\text{Re } (t) > 0$. This covers all physically interesting cases except $t = 0$.

Our method is to convert the determinant of the block matrix $M_n$ that arises in the dimer problem into a determinant of a block Toeplitz matrix and then to find a way to explicitly compute the asymptotics of the block Toeplitz determinant.

To be more specific the dimer matrix (1) has $n \times n$ matrix blocks $\mathcal{R}$ and $\mathcal{Q}$ whose entries are given by

$$\mathcal{R}_{jk} = 2(-1)^{(k-j)/2}R_{k-j+1} + \theta (j - k)t^{j-k-1},$$