On the Periodicity Conjecture for Y-systems

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Abstract: The conjecture in question (due ultimately to Alexei Zamolodchikov) asserts the periodicity of all the solutions to the so-called Y-systems. Those systems are naturally associated to pairs of indecomposable Cartan matrices of finite type, and the conjectured period is equal to twice the sum of the respective Coxeter numbers. This conjecture has so far been proven only if one of the ranks equals one, in which case the Y-systems are intrinsically related to Fomin-Zelevinsky’s cluster algebras. In this paper, I use elementary projective geometry to prove the case when the two Cartan matrices involved are of type A with both ranks arbitrary.

Introduction

Following Alexei Zamolodchikov [1], Kuniba and Nakanishi [2], and Ravanini et al [3], one can associate to any pair of indecomposable Cartan matrices of finite type the so-called Y-system of algebraic equations, which reads

\[ Y_{i_j+1,k} Y_{i_j-1,k} = \prod_{i' \neq i} (1 + Y_{i'jk})^{-a_{ii'}} \prod_{k' \neq k} (1 + 1/Y_{ijk'})^{-a_{kk'}} , \]

\[ i, j, k \in \mathbb{Z}, \quad 1 \leq i \leq r, \quad 1 \leq k \leq r', \]

where \((a_{ii'})\) and \((a_{kk'})\) are those Cartan matrices, and \(r\) and \(r'\) are the respective ranks. Then the periodicity conjecture asserts that all solutions to this system are periodic in \(j\), with period equal to twice the sum of the respective Coxeter numbers.

Until now this conjecture has only been proven in the case when one of the ranks equals one [4–6]. In this paper, we will take the next logical step and prove the case when the two Cartan matrices are of type A with both ranks arbitrary.
Recall that Cartan matrices of type A are tridiagonal, with twos on the diagonal and minus ones on the sub- and sup-diagonals. Thus, for $1 < i < r$ and $1 < k < r'$ we have

$$Y_{i,j+1}k Y_{i,j-1}k = \frac{(1 + Y_{i+1,j}k)(1 + Y_{i-1,j}k)}{(1 + 1/Y_{i,j+1}k)(1 + 1/Y_{i,j-1}k)},$$

while at the boundaries one or two factors in the right hand side are absent—for instance,

$$Y_{1,j+1}1 Y_{1,j-1}1 = \frac{1 + Y_{2,j}1}{1 + 1/Y_{1,j}1}.$$

However, they can be added back if we introduce fictitious boundary variables and set them equal to zero or infinity, as appropriate: $Y_{0j}k = Y_{r+1,j}k = 0$, $Y_{ij}0 = Y_{i,j,r+1} = \infty$. Note that $Y_{0j}0$, $Y_{0,j,r+1}$, $Y_{r+1,j}0$ and $Y_{r+1,j,r+1}$ are thus ill-defined, but they never appear in the right-hand side anyway.

Note also that our system consists of two completely decoupled identical subsystems—one involving variables $Y_{ijk}$ with $i + j + k$ even, and the other with $i + j + k$ odd. So we will simply discard the second subsystem and assume that the $Y_{ijk}$’s are only defined for $i + j + k$ even.

Finally, recall that the Coxeter number for type $A_r$ equals $r + 1$. Thus, we are going to prove the following:

**Theorem 1 (Periodicity conjecture, Case AA).** All regular solutions (that is, solutions avoiding 0, $-1$ and $\infty$ everywhere except the boundaries) to the Y-system

$$Y_{i,j+1}k Y_{i,j-1}k = \frac{(1 + Y_{i+1,j}k)(1 + Y_{i-1,j}k)}{(1 + 1/Y_{i,j+1}k)(1 + 1/Y_{i,j-1}k)},$$

where

$$Y_{0j}k = Y_{r+1,j}k = 0, \quad Y_{ij}0 = Y_{i,j,r+1} = \infty,$$

are $2(r + r' + 2)$-periodic in $j$:

$$Y_{i,j+2(r+r'+2),k} = Y_{ijk}.$$  

We will prove that by producing a manifestly periodic formula for the general solution. After the necessary preliminaries in Sect. 1, in Sect. 2 we check that this solution does actually satisfy the Y-system (0.2), and in Sect. 3 we complete the proof by showing that it is indeed a general solution.

Before we begin, though, a word of explanation is called for as to why such a magic formula should exist at all. Very roughly, this is because the Y-systems are closely related to the so-called Toda field theories. The simplest of those theories is the celebrated Liouville equation, and so our formula for solving the Y-system is basically a generalization of the Liouville formula for solving the Liouville equation. A more detailed explanation will be given elsewhere.

This paper is a slightly revised version of [7]. A very different proof of a closely related result appeared in [8]; see the closing remark of Sect. 2.