A Lower Bound for Nodal Count on Discrete and Metric Graphs

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Abstract: We study the number of nodal domains (maximal connected regions on which a function has constant sign) of the eigenfunctions of Schrödinger operators on graphs. Under a certain genericity condition, we show that the number of nodal domains of the \( n \)th eigenfunction is bounded below by \( n - \ell \), where \( \ell \) is the number of links that distinguish the graph from a tree.

Our results apply to operators on both discrete (combinatorial) and metric (quantum) graphs. They complement already known analogues of a result by Courant who proved the upper bound \( n \) for the number of nodal domains.

To illustrate that the genericity condition is essential we show that if it is dropped, the nodal count can fall arbitrarily far below the number of the corresponding eigenfunction.

In the Appendix we review the proof of the case \( \ell = 0 \) on metric trees which has been obtained by other authors.

1. Introduction

According to a well-know theorem by Sturm, the zeros of the \( n \)th eigenfunction of a vibrating string divide the string into \( n \) “nodal intervals”. The Courant nodal line theorem carries over one half of Sturm’s theorem to the theory of membranes: Courant proved that the \( n \)th eigenfunction cannot have more than \( n \) domains. He also provided an example showing that no non-trivial lower bound for the number of nodal domains can be hoped for in \( \mathbb{R}^d \), \( d \geq 2 \).

But what can be said about the number of nodal domains on graphs? Earliest research on graphs concentrated on Laplace and Schrödinger operators on discrete (combinatorial) graphs. The functions on discrete graphs take values on vertices of the graph and the Schrödinger operator is defined by

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\[(H\psi)_u = -\sum_{v \sim u} \psi_v + q_u \psi_u,\]

where the sum is taken over all vertices adjacent to the vertex \(u\).

Gantmacher and Krein [11] proved than on a chain graph (a tree with no branching which can be thought of as a discretization of the interval) an analogue of Sturm’s result holds: the \(n^{th}\) eigenvector changes sign exactly \(n - 1\) times. But for non-trivial graphs the situation departs dramatically from its \(\mathbb{R}^d\) analogue. First of all, Courant’s upper bound does not always hold. There is a correction due to multiplicity of the \(n^{th}\) eigenvalue and the upper bound becomes \([7]\) \(n + m - 1\), where \(m\) is the multiplicity. In this paper we discuss another striking difference. If the number of cycles of a graph is not large, the graph behaves “almost” like a string: for a typical eigenvector, there is a lower bound on the number of nodal domains.

To be more precise, let \(\ell\) be the minimal number of edges of the graph that distinguish it from a tree (a graph with no loops). In terms of the number of vertices \(V\) and the number of edges \(E\), the number \(\ell\) can be expressed as \(\ell = E - V + 1\). We show that, for a typical eigenvector, the number of nodal domains is greater or equal to \(n - \ell\). In particular, on trees (\(\ell = 0\)) the nodal counting is exact: the \(n^{th}\) eigenfunction has exactly \(n\) domains. Here by a “typical” eigenvector we mean an eigenvector which corresponds to a simple eigenvalue and which is not zero on any of the vertices. This property is stable with respect to small perturbations of the potential \(\{q_u\}\).

Another graph model on which the question of nodal domains is well-defined is the so-called quantum or metric graphs. These are graphs with edges parameterized by the distance to a pre-selected start vertex. The functions now live on the edges of the graph and are required to satisfy matching conditions on the vertices of the graph. The Laplacian in this case is the standard 1-dimensional Laplacian. A good review of the history of quantum graphs and some of their applications can be found in [17].

The ideas that the zeros of the eigenfunctions on the metric trees behave similarly to the 1-dimensional case have been around for some time. Al-Obeid, Pokorny and Pryadiev [1,21,20] showed that for a metric tree in a “general position” (which is roughly equivalent to our genericity assumption, see Sect. 3) the number of the nodal domains of the \(n^{th}\) eigenfunction is equal to \(n\). This result was rediscovered by Schapotschnikow [22] who was motivated by the recent interest towards nodal domains in the physics community [3,13,12].

Our result on the lower bound extends to the quantum graphs as well. Similarly to the discrete case, we prove that even for graphs with \(\ell > 0\), \(n - \ell\) is a lower bound on the number of nodal domains of the \(n^{th}\) eigenfunction.

The article is structured as follows. In Sect. 2 we explain the models we are considering, formulate our result and review the previous results on the nodal counting on graphs. The case of the metric trees has been treated before in [20,22]. In the three remaining cases, metric graphs with \(\ell > 0\), discrete trees and discrete graphs with \(\ell > 0\), we believe our results to be previously unknown and in Sect. 3 we provide complete proofs. For completeness, we also include a sketch of the general idea behind the proofs of [20,22] in the Appendix. Finally, in the last subsection of Sect. 3 we show that when a graph does not satisfy our genericity conditions, the nodal count can fall arbitrarily far below the number of the corresponding eigenfunction.

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1 We are talking here about the so-called “strong nodal domains” — maximal connected components on which the eigenfunction has a constant well-defined (i.e. not zero) sign.

2 Thus for a “typical” eigenvector the notions of “strong” and “weak” nodal domains (see [7]) coincide.