Yang-Mills Detour Complexes and Conformal Geometry

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Abstract: Working over a pseudo-Riemannian manifold, for each vector bundle with connection we construct a sequence of three differential operators which is a complex (termed a Yang-Mills detour complex) if and only if the connection satisfies the full Yang-Mills equations. A special case is a complex controlling the deformation theory of Yang-Mills connections. In the case of Riemannian signature the complex is elliptic. If the connection respects a metric on the bundle then the complex is formally self-adjoint. In dimension 4 the complex is conformally invariant and generalises, to the full Yang-Mills setting, the composition of (two operator) Yang-Mills complexes for (anti-)self-dual Yang-Mills connections. Via a prolonged system and tractor connection a diagram of differential operators is constructed which, when commutative, generates differential complexes of natural operators from the Yang-Mills detour complex. In dimension 4 this construction is conformally invariant and is used to yield two new sequences of conformal operators which are complexes if and only if the Bach tensor vanishes everywhere. In Riemannian signature these complexes are elliptic. In one case the first operator is the twistor operator and in the other sequence it is the operator for Einstein scales. The sequences are detour sequences associated to certain Bernstein-Gelfand-Gelfand sequences.

1. Introduction

In the study of Riemannian and pseudo-Riemannian geometry it is often valuable to use differential operators with good conformal behaviour. In the Riemannian setting, elliptic differential operators are particularly important. For example the conformal Laplacian controls the conformal variation of the scalar curvature. This was exploited heavily in the solution by Schoen, Aubin, Trudinger, and Yamabe (see \cite{40}) of the “Yamabe Problem” of finding, via conformal rescaling, constant scalar curvature metrics on compact manifolds. Related curvature prescription problems and techniques have exploited the higher order conformal Laplacians of Paneitz, Graham et al. \cite{8,18,35}. These operators...
on functions (or really densities) also find a natural place in the recent developments [24,36] concerning the asymptotics and scattering theory of the conformally compact Poincaré-Einstein metric of Fefferman-Graham [23].

On many tensor and spinor fields there is no conformally invariant elliptic operator (taking values in an irreducible bundle); this follows from the classification of conformally invariant differential operators on the sphere [7,22]. This classification is based on the structure of generalised Verma modules and from this it follows that often the analogue, or replacement, for a conformal elliptic operator on the sphere is an elliptic complex of conformally invariant differential operators. However the situation is complicated for conformally curved structures. The requirement that a sequence of differential operators be both conformally invariant and form a complex is severe. On the other hand when such complexes exist they can be expected to play a serious role in treating the underlying structure. This idea is already well-established in the setting of self-dual 4-manifolds [1,19]. On fully conformally curved $n$-manifolds, with $n$ even, there is a class of elliptic conformal complexes on differential forms [11]. Each of these is different to the de Rham complex, and these complexes generalise the conformally invariant operator of [35], with leading term $\Delta^{n/2}$. Another class of complexes is based around the (Fefferman-Graham) obstruction tensor [23]. This is a natural conformal 2-tensor that generalises, to higher even dimensions, the Bach tensor in dimension 4. It turns out that the formal deformations of obstruction-flat manifolds are controlled by a sequence of conformal operators, which form an elliptic complex if and only if the structure is obstruction-flat [12]. Unfortunately there is no obvious way to generalise either the construction in [11], or that in [12].

For 4-manifolds we construct here two conformal differential sequences which are (formally self-adjoint) complexes if and only if the (conformally invariant) Bach-tensor [2] vanishes everywhere. This condition is weaker than self-duality. In fact conformally Einstein manifolds are also Bach-flat and there are structures which are Bach-flat and neither conformally-Einstein nor half-flat [30]. Writing $T : S \to Tw$ for the usual twistor operator on Dirac spinors (as in e.g. [5]), in Theorem 4.5 we obtain a differential complex

$$S \xrightarrow{T} Tw \xrightarrow{M^\Sigma} Tw \xrightarrow{T^*} S,$$

where $M^\Sigma$ is a third order Rarita-Schwinger type operator. On the other hand in Theorem 4.3 we construct

$$E^0 \xrightarrow{P} E^{1,1} \xrightarrow{M^T} E^{1,1} \xrightarrow{P^*} E^0,$$

where $M^T$ is a second order conformal operator, similar in form to the operator which controls deformations of Einstein structures (see [6] and references therein), while $P$ is a curvature modification of the trace-free covariant Hessian. Non-vanishing solutions of $P$ give conformal factors $\sigma$ so that $\sigma^{-2}g$ is Einstein (see [3]); we show via the second sequence that the Bach tensor obstructs solutions. If the manifold is Riemannian then both of the complexes are elliptic. We have been intentionally explicit in treating these constructions, as it seems these complexes should play a fundamental role in conformal and Riemannian geometry. In the compact and Riemannian-signature setting the ellipticity implies that the complexes have finite dimensional cohomology spaces. In both cases the interpretation of the 0th-cohomology is well-known but as far as we know the first cohomology is a new global conformal invariant of Bach-flat structures.