Noncommutative Finite Dimensional Manifolds II: Moduli
Space and Structure of Noncommutative 3-Spheres

Alain Connes¹,²,³, Michel Dubois-Violette⁴

¹ Collège de France, 3, rue d’Ulm, Paris, F-75005 France. E-mail: alain@connes.org
² I.H.E.S., 35, route deChartres, F-91440 Bures-sur-Yvette, France
³ Mathematics Department, Vanderbilt University, Nashville, TN 37235, USA
⁴ Laboratoire de Physique Théorique, UMR 8627, Université Paris XI, Bâtiment 210,
  F-91 405 Orsay Cedex, France. E-mail: Michel.Dubois-Violette@th.u-psud.fr

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Abstract: This paper contains detailed proofs of our results on the moduli space and
the structure of noncommutative 3-spheres. We develop the notion of central quadratic
form for quadratic algebras, and a general theory which creates a bridge between non-
commutative differential geometry and its purely algebraic counterpart. It allows to
construct a morphism from an involutive quadratic algebra to a C*-algebra constructed
from the characteristic variety and the hermitian line bundle associated to the central
quadratic form. We apply the general theory in the case of noncommutative 3-spheres
and show that the above morphism corresponds to a natural ramified covering by a non-
commutative 3-dimensional nilmanifold. We then compute the Jacobian of the ramified
covering and obtain the answer as the product of a period (of an elliptic integral) by a
rational function. We describe the real and complex moduli spaces of noncommutative
3-spheres, relate the real one to root systems and the complex one to the orbits of a
birational cubic automorphism of three dimensional projective space. We classify the
algebras and establish duality relations between them.

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