Conformal Operators on Forms and Detour Complexes on Einstein Manifolds

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Abstract: For even dimensional conformal manifolds several new conformally invariant objects were found recently: invariant differential complexes related to, but distinct from, the de Rham complex (these are elliptic in the case of Riemannian signature); the cohomology spaces of these; conformally stable form spaces that we may view as spaces of conformal harmonics; operators that generalise Branson’s Q-curvature; global pairings between differential form bundles that descend to cohomology pairings. Here we show that these operators, spaces, and the theory underlying them, simplify significantly on conformally Einstein manifolds. We give explicit formulae for all the operators concerned. The null spaces for these, the conformal harmonics, and the cohomology spaces are expressed explicitly in terms of direct sums of subspaces of eigenspaces of the form Laplacian. For the case of non-Ricci flat spaces this applies in all signatures and without topological restrictions. In the case of Riemannian signature and compact manifolds, this leads to new results on the global invariant pairings, including for the integral of Q-curvature against the null space of the dimensional order conformal Laplacian of Graham et al.

1. Introduction

Differential forms provide a fundamental domain for the study of smooth manifolds. In Riemannian geometry the de Rham complex, its associated Hodge theory, and distinguished forms representing characteristic classes are among the most basic and important tools (e.g. [14,15]). In physics the study of forms is partly motivated by Maxwell theory and its generalisations. Operators on differential forms feature strongly in string and brane theories. In both mathematics and physics Einstein manifolds have a central position [2] and thus they give an important class of special structures for the study of geometric objects.

Among the differential operators that are natural for pseudo–Riemannian structures only a select class are conformally invariant. Conformal invariance is a subtle
property which reflects an independence of the point dependent scale. This symmetry is manifest in the equations of massless particles. It is linked to CR geometry (hence complex analysis) through the Fefferman metric \[19\]; the natural equations on the Fefferman space are conformally invariant. This symmetry also underpins the conformal approach to Riemannian geometry. For example, it is essentially exploited in the Yamabe problem (see \[36\] and references therein) of prescribing the scalar curvature. Recently there has been a focus on variations of this idea, including the conformal prescription of Branson’s Q-curvature \[5,13,16\]. These problems use the conformal Laplacian on functions (or densities) and its higher order analogues due to Paneitz, Graham et al. \[32\].

The use of conformal operators on forms provides a setting where, on the one hand, there is potential to formally generalise such theories, but which, on the other hand, should yield access to rather different geometric data. An immediate difficulty is that forms are more difficult to work with than functions and so, while there was much early work in this direction (e.g. \[3,4\]), this did not yield a clear picture. In dimension 4, and inspired by constructions from twistor theory, some rather interesting directions and applications to global geometry were pioneered in the work of Eastwood and Singer \[17,18\]. Links between this result and the tractor calculus of \[1,9,10\] were established in \[6\]. On the other hand in \[11,27\] it is shown that the conformal tractor connection may be recovered as a suitable linearisation of the ambient metric of Fefferman and Graham \[20\] (and see also \[21\]). Exploiting both developments a rather complete theory of conformal operators on forms was derived in the joint works \[7,8\] of the first author with Branson. The main point of these articles was not simply to construct conformal operators on differential forms, but rather, to expose and develop the discovery of preferred versions of such operators and the rather elegant picture that these yield: one may immediately construct, on even dimensional conformal manifolds, a host of new global conformal invariants. Some of these generalise, in a natural way, the integral of Q-curvature.

For most of these new operators explicit formulae are not available. For any particular operator a formula may be obtained algorithmically via tractor calculus and the theory developed in \[27,28\]. However the resulting operators, when presented in the usual way, are given by extremely complicated formulae. It turns out there are striking simplifications when these operators are studied on conformally Einstein manifolds. The purpose of this article is to expose this, via a comprehensive but concise treatment, and use the results to study, in the Einstein setting, the related global conformal invariants and spaces.

To describe the content in more detail we first review the relevant results from \[7\] and \[8\]. On conformal manifolds of even dimension \(n \geq 4\) there is a family of formally self-adjoint conformally invariant differential complexes:

\[
\mathcal{E}^0 \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{E}^{k-1} \xrightarrow{d} \mathcal{E}^k \xrightarrow{L_k} \mathcal{E}_k \xrightarrow{\delta} \mathcal{E}_{k-1} \xrightarrow{\delta} \ldots \xrightarrow{\delta} \mathcal{E}_0.
\]  

(1)

Here, for each \(k \in \{0, 1, \ldots, n/2 - 1\}\), \(\mathcal{E}^k\) denotes the space of \(k\)-forms, \(\mathcal{E}_k\) denotes an appropriate density twisting of that space, \(d\) is the exterior derivative and \(\delta\) its formal adjoint. An interesting feature of these complexes is that the operators \(L_k\) have the structure of a composition

\[
L_k = \delta Q_{k+1} d,
\]

where \(Q_{k+1}\) is from a family of differential operators, parametrised by \(k = -1, \ldots, n/2-1\), and which, as operators on closed forms, generalise Branson’s Q-curvature; in particular under conformal rescaling of the metric \(g \mapsto \tilde{g} = e^{2\omega} g (\omega \in C^\infty)\) these have the