Reflectionless Herglotz Functions and Jacobi Matrices

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Abstract: We study several related aspects of reflectionless Jacobi matrices. First, we discuss the singular part of the corresponding spectral measures. We then show how to identify sets on which measures are reflectionless by looking at the logarithmic potentials of these measures.

1. Introduction

We study several aspects of reflectionless Jacobi matrices and Herglotz functions in this paper. This is part of a larger program; the (perhaps too ambitious) goal is to reach a systematic understanding of the absolutely continuous spectrum of Jacobi operators $J$ on $\ell_2(\mathbb{Z}_+)$,

$$(J u)(n) = a(n)u(n+1) + a(n-1)u(n-1) + b(n)u(n).$$

We will always assume that the coefficients $a$, $b$ satisfy bounds of the form

$$(C + 1)^{-1} \leq a(n) \leq C + 1, \quad |b(n)| \leq C,$$

for some $C > 0$. Note that if $J$ has some absolutely continuous spectrum, then, by the decoupling argument of Dombrowski and Simon-Spencer [3,17], it actually suffices to assume that $a(n)$ is bounded above; the other two inequalities follow automatically.

Let us recall some definitions. A Herglotz function is a holomorphic mapping of $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im} z > 0\}$ to itself. We denote the set of Herglotz functions by $\mathcal{H}$. If $F \in \mathcal{H}$, then $F(t) \equiv \lim_{y \to 0^+} F(t + iy)$ exists for (Lebesgue) almost every $t \in \mathbb{R}$. We call $F$ reflectionless (on $E \subset \mathbb{R}$) if

$$\text{Re } F(t) = 0 \quad \text{for almost every } t \in E.$$  

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We will also use the notation
\[ \mathcal{N}(E) = \{ F \in \mathcal{H} : F \text{ reflectionless on } E \}. \]

Herglotz functions have unique representations of the form
\[
F(z) = F_\mu(z) = a + bz + \int_{-\infty}^{\infty} \left( \frac{1}{t-z} - \frac{t}{t^2 + 1} \right) d\mu(t),
\]
with \( a \in \mathbb{R}, b \geq 0, \) and a (positive) Borel measure \( \mu \) on \( \mathbb{R} \), \( \int_{\mathbb{R}} \frac{d\mu(t)}{t^2 + 1} < \infty \). We will call such a measure \( \mu \) reflectionless (on \( E \)) if \( F_\mu \in \mathcal{N}(E) \) for some choice of \( a, b \geq 0 \); for easier reference, it will also be convenient to introduce the notation
\[ \mathcal{R}(E) = \{ \mu : \mu \text{ reflectionless on } E \}. \]

We emphasize again that in particular \( \int_{\mathbb{R}} \frac{d\mu(t)}{t^2 + 1} < \infty \) for all \( \mu \in \mathcal{R}(E) \). Also, if \( \mu \in \mathcal{R}(E) \), then \( F_\mu \) will refer to the unique Herglotz function \( F_\mu \in \mathcal{N}(E) \) that is associated with \( \mu \) as in (1.2).

There are several reasons for being interested in the class \( \mathcal{N}(E) \); here, our main motivation is provided by the following fact: Call a (whole line) Jacobi matrix \( J \) reflectionless (on \( E \)) if \( g_n \in \mathcal{N}(E) \) for all \( n \in \mathbb{Z} \), where \( g_n(z) = \langle \delta_n, (J-z)^{-1}\delta_n \rangle \) is the \( n \)th diagonal element of the resolvent of \( J \) (also known as the Green function). Then [13, Theorem 1.4] says that all \( \omega \) limit points of a Jacobi matrix \( J \) with some absolutely continuous spectrum are reflectionless on \( E = \Sigma_{ac} \); here, \( \Sigma_{ac} \) denotes an essential support of the absolutely continuous part of the spectral measure \( \rho \) of \( J \). This is defined up to sets of Lebesgue measure zero; we can obtain a representative as \( \Sigma_{ac} = \{ t : d\rho/dt > 0 \} \). Please see [13] for the details.

If \( \mu \in \mathcal{R}(E) \), then \( \chi_E dt \ll d\mu_{ac} \). Indeed, this follows immediately from (1.1) because \( d\mu_{ac}(t) = (1/\pi)\text{Im } F(t) dt \) and the boundary value of a Herglotz function can not be zero on a set of positive measure. However, it is not so clear in general if \( \mu \) can also have a singular part on \( E \). See [4–6] for earlier work on this question. We have the following criterion. We say that a (positive) measure \( \nu \) is supported by a (measurable) set \( S \) if \( \nu(S^c) = 0 \); supports are not assumed to be closed in this paper.

**Theorem 1.1.** Let \( \mu \in \mathcal{R}(E) \). Then:

(a) \( \mu_s \), the singular part of \( \mu \), is supported by
\[
\left\{ x \in \mathbb{R} : \lim_{h \to 0^+} \frac{|E \cap (x-h, x+h)|}{2h} = 0 \right\}.
\]

(b) Let \( \theta \in L_\infty(E) \) be an arbitrary bounded measurable function. Then \( \mu_s \) is also supported by
\[
\{ x \in \mathbb{R} : (\tilde{H}_E \theta)(x) \text{ exists} \}.
\]