Stein’s Method and Characters of Compact Lie Groups

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Abstract: Stein’s method is used to study the trace of a random element from a compact Lie group or symmetric space. Central limit theorems are proved using very little information: character values on a single element and the decomposition of the square of the trace into irreducible components. This is illustrated for Lie groups of classical type and Dyson’s circular ensembles. The approach in this paper will be useful for the study of higher dimensional characters, where normal approximations need not hold.

1. Introduction

There is a large literature on the traces of random elements of compact Lie groups. One of the earliest results is due to Diaconis and Shahshahani [DS]. Using the method of moments, they show that if $g$ is random from the Haar measure of the unitary group $U(n, \mathbb{C})$, and $Z = X + iY$ is a standard complex normal with $X$ and $Y$ independent, mean 0 and variance $\frac{1}{2}$ normal variables, then for $j = 1, 2, \ldots, Tr(g^j)$ are independent and distributed as $\sqrt{j}Z$ asymptotically as $n \to \infty$. They give similar results for the orthogonal group $O(n, \mathbb{R})$ and the group of unitary symplectic matrices $USp(2n, \mathbb{C})$. The moment computations of [DS] use representation theory. It is worth noting that there are other approaches to their moment computations: [PV] uses a version of integration by parts (and also treats $SO(n, \mathbb{R})$), and [CoSz] uses an “extended Wick calculus” (and also treats symmetric spaces).

Concerning the error in the normal approximation in the [DS] results, Diaconis conjectured that for fixed $j$, it decreases exponentially or even superexponentially in $n$. Stein [St2] uses “Stein’s method” to show that $Tr(g^k)$ on $O(n, \mathbb{R})$ is asymptotically normal with error $O(n^{-r})$ for any fixed $r$. Johansson [J] proved Diaconis’ conjecture for classical compact Lie groups using Toeplitz determinants and a very detailed analysis of characteristic functions.

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One direction in which the [DS] results have been extended is the study of linear statistics of eigenvalues: see [J,DE,So] and the numerous references therein. There is also work by D’Aristotle, Diaconis, and Newman [DDN] on central limit theorems for linear functions such as \( Tr(Ag) \), where \( A \) is a fixed \( n \times n \) real matrix and \( g \) is from the Haar measure of \( O(n, \mathbb{R}) \). In recent work, Meckes [Me2] refined Stein’s technique from [St2] to establish a sharp total variation distance error term (order \( n^{-1} \)) for the [DDN] result.

A natural goal is to prove limit theorems (with error terms) for the distribution of traces in other irreducible representations: i.e. \( \chi^\tau(g) \), where \( g \) is a random element of a compact Lie group and \( \chi^\tau \) is the character of an irreducible representation \( \tau \). This would have direct implications for Katz’s work [Ka] on exponential sums; see Sect. 4.7 of [KLR] for details. We do not attain this goal, but make a useful contribution to it.

More precisely, the current paper presents a formulation of Stein’s method designed for the study of \( \chi^\tau(g) \). In the case of normal approximation, we obtain \( O(n^{-1}) \) bounds for the error term using only two pieces of information:

- The value of the “character ratios” \( \frac{\chi^\phi(\alpha)}{\dim(\phi)} \), where \( \phi \) may be arbitrary but \( \alpha \) is a single element of \( G \) (typically chosen to be close to the identity)
- The decomposition of \( \tau \) into irreducible representations.

In contrast, the method of moments approach requires knowing the multiplicity of the trivial representation in \( \tau^k \) for all \( k \geq 1 \) (which could be tricky to compute) and does not give an immediate bound on the error. Johansson’s paper [J] gives sharper bounds when \( \chi^\tau \) is the trace of an element from a classical compact Lie group, but requires knowledge of high order moments and deep analytical tools which might not extend to arbitrary representations \( \tau \). Even Stein’s method approaches of Stein [St2] and Meckes [Me2] use information about the distribution of matrix entries; very little is known about this for arbitrary \( \tau \), whereas the main ingredient for our approach (character theory) is well-developed.

Let us explain our statement in the abstract that the methods of this paper will prove useful for approximation other than normal approximation. We use Stein’s method of exchangeable pairs which involves the construction of a pair \((W, W')\) of exchangeable random variables. Our pair (which is somewhat different from those of Stein [St2] and Meckes [Me2]) satisfies the linearity condition that \( \mathbb{E}(W' | W) \) is proportional to \( W \), and we find representation theoretic formulas for quantities such as \( \mathbb{E}(W' - W)^k \). These computations are completely general and apply to arbitrary distributional approximation. Stein’s method of exchangeable pairs is still quite undeveloped for continuous distributions other than the normal, but that is temporary and there are some results: see [Mn,Re] for the chi-squared distribution, [Lu] for the Gamma distribution, and [GoT] for the semicircle law. Closest to the current paper is [CFR], which develops error terms for exponential approximation using quantities like \( \mathbb{E}(W' - W)^k \) with \( k \) small.

We remark that the bounds in our paper are all given in the Kolmogorov metric. Similar results can be proved in the slightly stronger total variation metric (see the remarks after Theorem 2.1). However we prefer to work in the Kolmogorov metric as it underscores the similarity with discrete settings such as [Fu], where total variation convergence does not occur. We also mention that all bounds obtained in this paper are given with explicit constants.

The organization of this paper is as follows. Section 2 gives background on Stein’s method and normal approximation. Section 3 develops general theory for the case that \( G \) is a compact Lie group and \( \chi^\tau \) an irreducible character. It treats the trace of random