Hilbert-Schmidt Operators vs. Integrable Systems of Elliptic Calogero-Moser Type I. The Eigenfunction Identities

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Abstract: In this series of papers we study Hilbert-Schmidt integral operators acting on the Hilbert spaces associated with elliptic Calogero-Moser type Hamiltonians. As shown in this first part, the integral kernels are joint eigenfunctions of differences of the latter Hamiltonians. On the relativistic (difference operator) level the kernel is built from the elliptic gamma function, whereas the building block in the nonrelativistic (differential operator) limit is basically the Weierstrass sigma-function. For the \( A_{N-1} \) case we consider all of the commuting Hamiltonians at once, the eigenfunction properties reducing to a sequence of elliptic identities. For the \( BC_N \) case we only treat the defining Hamiltonians. The functional identities encoding the eigenfunction properties have a remarkable corollary in the relativistic \( BC_1 \) case: They imply that the sum over eight-fold products of the four Jacobi theta functions is invariant under the Weyl group of \( E_8 \).

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1. Introduction

The nonrelativistic elliptic Calogero-Moser Hamiltonian is given by the PDO (partial differential operator)
\[ H_{\text{nr}}(x) = -\frac{1}{2} \sum_{j=1}^{N} \partial_{x_j}^2 + g(g-1) \sum_{1 \leq j < k \leq N} \wp(x_j - x_k; \pi/2r, i\alpha/2), \quad g \in \mathbb{R}, \quad r, \alpha > 0. \]

(1.1)

Here, the pair potential is the Weierstrass \( \wp \)-function; its periods \( \pi/r \) and \( i\alpha \) are chosen positive and purely imaginary, so that the PDO is formally self-adjoint. As is well known, this Hamiltonian defines a quantum integrable system: There exist \( N \) commuting PDOs

\[ H_1 = -i \sum_{j=1}^{N} \partial_{x_j}, \quad H_2 = H_{\text{nr}}, \quad H_k = \frac{(-i)^k}{k} \sum_{j=1}^{N} \partial_{x_j}^k + \text{l. o.}, \quad k = 3, \ldots, N, \]

(1.2)

where l. o. denotes a PDO of lower order in the partials with elliptic coefficients [1–3]. To date, however, no joint eigenfunctions are known to exist for arbitrary \( g, r, \alpha > 0 \), save for \( N = 2 \); in that case, the Schrödinger equation for (1.1) amounts to the Lamé equation.

The \( N \) commuting Hamiltonians of the relativistic Calogero-Moser system can be chosen to be the A\( \Delta \)Os (analytic difference operators)

\[ S_l(x) = \prod_{\substack{I \subset \{1, \ldots, N\} \atop |I| = l}} f_-(x_j - x_k) \exp(-i\beta \sum_{j \in I} \partial_{x_j}) \prod_{\substack{I \subset \{1, \ldots, N\} \atop |I| = l}} f_+(x_j - x_k), \]

\[ l = 1, \ldots, N, \quad \beta > 0, \]

(1.3)

where

\[ f_{\pm}(z) = [\sigma(z \pm i\beta g; \pi/2r, i\alpha/2)/\sigma(z; \pi/2r, i\alpha/2)]^{1/2}, \]

(1.4)

with \( \sigma(z) \) the Weierstrass \( \sigma \)-function, cf. [4,5]. In this case too, the existence of joint eigenfunctions for \( N > 2 \) and arbitrary parameters is not known.

The A\( \Delta \)Os \( S_{\pm l} \) obtained from (1.3) by switching \( f_- \) and \( f_+ \) and taking \(-i \to i\) in the exponential commute as well, and moreover commute with \( S_1, \ldots, S_N \). Furthermore, when one interchanges the positive parameters \( \alpha \) and \( \beta \), one obtains yet another set of \( 2N \) mutually commuting A\( \Delta \)Os that also commute with the \( 2N \) A\( \Delta \)Os \( S_{\pm l} \). To handle all of these operators simultaneously, it is expedient to replace \( \alpha \) and \( \beta \) by \( a_+ \) and \( a_- \), cf. Sect. 2.

The above integrable systems are associated with the root system \( A_{N-1} \), and they have versions for other root systems as well. The \( BC_N \) version of the nonrelativistic Calogero-Moser system is defined by the Inozemtsev Hamiltonian [6]

\[ H_{\text{nr}}(g, \lambda; x) \equiv -\sum_{j=1}^{N} \partial_{x_j}^2 + 2\lambda(\lambda - 1) \sum_{1 \leq j < k \leq N} \wp(x_j - \delta x_k) \]
\[ + \sum_{t=0}^{3} \sum_{j=1}^{N} g_t(g_t - 1) \wp(x_j + \omega_t). \]

(1.5)

The existence of \( N-1 \) commuting PDOs of higher order follows from papers by Oshima and coworkers [2,3]. For \( N = 1 \) the Schrödinger equation for (1.5) reduces to the Heun