Bethe Algebra of Homogeneous XXX Heisenberg Model has Simple Spectrum

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Abstract: We show that the algebra of commuting Hamiltonians of the homogeneous XXX Heisenberg model has simple spectrum on the subspace of singular vectors of the tensor product of two-dimensional \( \mathfrak{gl}_2 \)-modules. As a byproduct we show that there exist exactly \( \binom{n}{l} - \binom{n}{l-1} \) two-dimensional vector subspaces \( V \subset \mathbb{C}[u] \) with a basis \( f, g \in V \) such that \( \text{deg } f = l, \text{deg } g = n - l + 1 \) and \( f(u)g(u-1) - f(u-1)g(u) = (u + 1)^n \).

1. Introduction

1.1. Homogeneous XXX Heisenberg model. Consider the vector space \( (\mathbb{C}^2)^{\otimes n} \) and the linear operator

\[
H_{xxx} = -\sum_{j=1}^{n} \left( \sigma_1^{(j)} \sigma_1^{(j+1)} + \sigma_2^{(j)} \sigma_2^{(j+1)} + \sigma_3^{(j)} \sigma_3^{(j+1)} \right),
\]

where \( \sigma_a^{(k)} = 1^{\otimes (k-1)} \otimes \sigma_a \otimes 1^{\otimes (n-k)} \), \( \sigma_a^{(n+1)} = \sigma_a^{(1)} \), and \( \sigma_1, \sigma_2, \sigma_3 \) are the Pauli matrices,

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The operator \( H_{xxx} \) is the Hamiltonian of the celebrated XXX Heisenberg model, also called the homogeneous XXX model, and the problem is to find eigenvalues and eigenvectors of the Hamiltonian.

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This problem was first addressed in the pioneering work \[Be\] by H. Bethe, who looked for eigenvectors of \(H_{xxx}\) in a certain special form. His method and its further extensions are traditionally called the Bethe ansatz. The current literature on the XXX model and its generalizations, XXZ and XYZ models, as well as their counterparts in statistical mechanics, the six- and eight-vertex models, is enormous. We limit ourselves to mentioning just two books, [B1] and [KBI]. However, even numerous references therein hardly cover half of the bibliography on the subject.

The Hamiltonian \(H_{xxx}\) can be included into a one-parameter family of commuting linear operators called the transfer matrix, see \[B1,FT,KBI\]. We call a commutative unital subalgebra of linear operators on \((\mathbb{C}^2)^{\otimes n}\) generated by the transfer matrix the Bethe algebra. The actual problem is to construct eigenvalues and eigenvectors for the Bethe algebra.

The elements of the Bethe algebra commute with the natural \(gl_2\)-action on \((\mathbb{C}^2)^{\otimes n}\). Therefore, the eigenspaces of the Bethe algebra are representations of \(gl_2\), and it suffices to construct highest weight vectors of those representations.

The Bethe ansatz method associates to every admissible solution \((\lambda_1, \ldots, \lambda_l)\) of the system of equations

\[
\left( \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}} \right)^n \prod_{\substack{k=1 \atop k \neq j}}^l \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}, \quad j = 1, \ldots, l, \tag{1.1}
\]

a vector in \((\mathbb{C}^2)^{\otimes n}\), called the corresponding Bethe vector, see [FT]. A solution \((\lambda_1, \ldots, \lambda_l)\) is called admissible if all \(\lambda_1, \ldots, \lambda_l\) are distinct, and all factors in (1.1) are nonzero. A nonzero Bethe vector is a highest weight vector of an \((n - 2l + 1)\)-dimensional irreducible representation of \(gl_2\), and all vectors in that representation are eigenvectors of each element of the Bethe algebra sharing the same eigenvalue.

It is an important question whether the Bethe ansatz method produces all eigenvectors of the Bethe algebra. This question is referred to as the question of completeness of the Bethe ansatz for finite chains. It was discussed by H. Bethe himself in \[Be\] and many times since then by other authors. For instance, see a recent discussion in \[B2\]. However, no rigorous proof is available even for the so-called inhomogeneous models. Moreover, as one can see from the results of this paper, Sklyanin’s separation of variables does not prove completeness of the Bethe ansatz to the very end, though it is indeed an important step towards the proof.

To be more precise, there are certain quantum integrable models for which the completeness of the Bethe ansatz has been proved. For example, see [YY] and Theorem 1.2.2 in [KBI]. The proofs for those models are based on a variational principle and convexity of some auxiliary action. However, for the the XXX model, the corresponding action is not convex, and that technique fails.

In this paper we establish the completeness of the Bethe ansatz method for the homogeneous XXX model provided the method is improved in a certain way, see below in the introduction. We show that the spectrum of the Bethe algebra of the homogeneous XXX model is simple, that is, all eigenspaces of the Bethe algebra are irreducible \(gl_2\)-modules. We also show that eigenvalues of the Bethe algebra are in a one-to-one correspondence with certain second-order linear difference equations with two linearly independent polynomial solutions. We prove similar results for inhomogeneous higher spin XXX models.