Equivariant Differential Characters and Symplectic Reduction

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Abstract: We describe equivariant differential characters (classifying equivariant circle bundles with connections), their prequantization, and reduction.

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0. Introduction

This paper is a continuation of [11]. The goal of the project is to understand geometric (pre)quantization and symplectic reduction in the context of stacks. There are two practical reasons for doing that. First, even if one is only interested in global Lie group actions on manifolds, symplectic quotients are generally orbifolds (i.e. Deligne-Mumford stacks). Second, the language of stacks allows one to work locally and thus to avoid messy Čech-type arguments.

0.1. Prequantization. The classic geometric prequantization theorem (due to Weil [16] and Kostant [10]) says that given a closed integral differential 2-form $\omega$ on a manifold $M$ there exists a principal $S^1$-bundle $P \to M$ with a connection $A$ such that the curvature of $A$ is equal to $\omega$, and moreover, the set of isomorphism classes of such pairs $(P, A)$ is a principal homogeneous space of the group of flat $S^1$-bundles. Recall that a 2-form $\omega$ on a manifold $M$ is integral if $\int_S \omega \in \mathbb{Z}$ for any closed smooth singular 2-chain $S \in Z_2(M)$.

It would be preferable for prequantization procedure to produce unique output (an $S^1$-bundle with connection). For example, this is clearly required to make sense of statements like “prequantization commutes with reduction” - see below. In order to have unique output of the prequantization one has to refine its input. One way to do it, due to Cheeger and Simons [5], is by using differential characters (an alternative approach is provided by Deligne cohomology - see [3]). A differential character of degree 2 (degree 1 in Cheeger-Simons grading) is a pair $(\omega, \chi)$, where $\omega \in \Omega^2(M)$ is a closed 2-form and $\chi : Z_1(M) \to \mathbb{R}/\mathbb{Z}$ is a character of the group $Z_1(M)$ of smooth singular 1-cycles. This pair should satisfy the following compatibility condition:

$$\chi(\partial S) = \int_S \omega \mod \mathbb{Z},$$

for any smooth singular 2-chain $S \in C_2(M)$. One can show (cf. [5, 9, 11] for various versions of the proof) that differential characters classify isomorphism classes of principal $S^1$-bundles with connections. The classifying bijection (whose inverse is the prequantization map) associates to a principal $S^1$-bundle $P$ with a connection $A$ its differential character $(\omega, \chi)$, where $\omega$ is the curvature of $A$ and $\chi$ is the holonomy of $A$ (we identify $S^1$ with $\mathbb{R}/\mathbb{Z}$ throughout the paper).