TYZ Expansion for the Kepler Manifold

Todor Gramchev, Andrea Loi

Dipartimento di Matematica e Informatica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy.
E-mail: todor@unica.it; loi@unica.it

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Abstract: The main goal of the paper is to address the issue of the existence of Kempf’s distortion function and the Tian-Yau-Zelditch (TYZ) asymptotic expansion for the Kepler manifold - an important example of non-compact manifold. Motivated by the recent results for compact manifolds we construct Kempf’s distortion function and derive a precise TYZ asymptotic expansion for the Kepler manifold. We get an exact formula: finite asymptotic expansion of \( n - 1 \) terms and exponentially small error terms uniformly with respect to the discrete quantization parameter \( m \to \infty \) (\( \hbar = m^{-1} \to 0 \) standing for Planck’s constant and \( |x| \to \infty \), \( x \in \mathbb{C}^n \)). Moreover, the coefficients are calculated explicitly and they turned out to be homogeneous functions with respect to the polar radius in the Kepler manifold. We show that our estimates are sharp by analyzing the nonharmonic behaviour of \( T_m \) for \( m \to +\infty \). The arguments of the proofs combine geometrical methods, quantization tools and functional analytic techniques for investigating asymptotic expansions in the framework of analytic-Gevrey spaces.

1. Introduction and Statements of the Main Results

Let \( g \) be a Kähler metric on an \( n \)-dimensional complex manifold \( M \). Assume that \( g \) is polarized with respect to a holomorphic line bundle \( L \) over \( M \), i.e. \( c_1(L) = [\omega] \), where \( \omega \) is the Kähler form associated to \( g \) and \( c_1(L) \) denotes the first Chern class of \( L \). Let \( m \geq 1 \) be a non-negative integer and let \( h_m \) be a Hermitian metric on \( L^m = L^{\otimes m} \) such that its Ricci curvature \( \text{Ric}(h_m) = m\omega \). Here \( \text{Ric}(h_m) \) is the two–form on \( M \) whose local expression is given by

\[
\text{Ric}(h_m) = -\frac{i}{2} \partial \bar{\partial} \log h_m(\sigma(x), \sigma(x)),
\]

(1.1)

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for a trivializing holomorphic section $\sigma : U \to L^m \setminus \{0\}$. In the quantum mechanics terminology $L^m$ is called the *quantum line bundle*, the pair $(L^m, h_m)$ is called a *geometric quantization* of the Kähler manifold $(M, m\omega)$ and $\hbar = m^{-1}$ plays the role of Planck’s constant (see e.g. [2]). Consider the separable complex Hilbert space $\mathcal{H}_m$ consisting of global holomorphic sections $s$ of $L^m$ such that

$$\langle s, s \rangle_m = \int_M h_m(s(x), s(x)) \frac{\omega^n}{n!} < \infty.$$  

Let $x \in M$ and let $q \in L^m \setminus \{0\}$ be a fixed point of the fiber over $x$. If one evaluates $s \in \mathcal{H}_m$ at $x$, one gets a multiple $\delta_q(s)$ of $q$, i.e. $s(x) = \delta_q(s)q$. The map $\delta_q : \mathcal{H}_m \to \mathbb{C}$ is a continuous linear functional [9]. Hence from Riesz’s theorem, there exists a unique $e^m_q \in \mathcal{H}$ such that $\delta_q(s) = \langle s, e^m_q \rangle_m, \forall s \in \mathcal{H}_m$, i.e.

$$s(x) = \langle s, e^m_q \rangle_m q. \quad (1.2)$$

It follows that $e^m_q = \overline{e}^{-1} e^m_q, \forall c \in \mathbb{C}^*$. The holomorphic section $e^m_q \in \mathcal{H}_m$ is called the *coherent state* relative to the point $q$. Thus, one can define a smooth function on $M$,

$$T_m(x) = h_m(q, q) \| e^m_q \|^2, \quad \| e^m_q \|^2 = \langle e^m_q, e^m_q \rangle,$$  

where $q \in L^m \setminus \{0\}$ is any point on the fiber of $x$. If $j, j = 0, \ldots, d_m, (d_m + 1 = \dim \mathcal{H}_m \leq \infty)$ form an orthonormal basis for $(\mathcal{H}_m, \langle \cdot, \cdot \rangle_m)$ then one can easily verify that

$$T_m(x) = \sum_{j=0}^{d_m} h_m(s_j(x), s_j(x)). \quad (1.4)$$

Notice that when $M$ is compact $\mathcal{H}_m = H^0(L^m)$, where $H^0(L^m)$ denotes the space of global holomorphic sections of $L^m$. Hence in this case $d_m < \infty$ and (1.4) is a finite sum.

The function $T_m$ has appeared in the literature under different names. The earliest one was probably the $\eta$-function of J. Rawnsley [42] (later renamed to $\epsilon$ function in [9]), defined for arbitrary Kähler manifolds, followed by the *distortion function* of Kempf [27] and Ji [26], for the special case of Abelian varieties and of Zhang [48] for complex projective varieties. The metrics for which $T_m$ is constant were called *critical* in [48] and *balanced* in [17] (see also [3,32 and 33]). If $T_m$ are constants for all sufficiently large $m$ then the geometric quantization $(L^m, h_m)$ associated to the Kähler manifold $(M, g)$ is called regular. Regular quantization plays a prominent role in the theory of quantization by deformation of Kähler manifolds developed in [9].

Fix $m \geq 1$. Under the hypothesis that for each point $x \in M$ there exists $s \in \mathcal{H}_m$ non-vanishing at $x$, one can give a geometric interpretation of $T_m$ as follows. Consider the holomorphic map of $M$ into the complex projective space $\mathbb{C} P^{d_m}$:

$$\varphi_m : M \to \mathbb{C} P^{d_m} : x \mapsto [s_0(x) : \cdots : s_{d_m}(x)]. \quad (1.5)$$

One can prove that

$$\varphi^*_m(\omega_{FS}) = m\omega + \frac{i}{2} \partial \overline{\partial} \log T_m, \quad (1.6)$$