A Criterion for Uniqueness of Lagrangian Trajectories for Weak Solutions of the 3D Navier-Stokes Equations

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Received: 7 July 2008 / Accepted: 17 February 2009
Published online: 19 May 2009 – © Springer-Verlag 2009

Abstract: Foias, Guillopé, & Temam showed in 1985 that for a given weak solution $u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ of the three-dimensional Navier-Stokes equations on a domain $\Omega$, one can define a ‘trajectory mapping’ $\Phi : \Omega \times [0, T] \to \Omega$ that gives a consistent choice of trajectory through each initial condition $a \in \Omega$, $\xi_a(t) = \Phi(a, t)$, and that respects the volume-preserving property one would expect for smooth flows. The uniqueness of this mapping is guaranteed by the theory of renormalised solutions of non-smooth ODEs due to DiPerna & Lions.

However, this is a distinct question from the uniqueness of individual particle trajectories. We show here that if one assumes a little more regularity for $u$ than is known to be the case, namely that $u \in L^{6/5}(0, T; L^\infty(\Omega))$, then the particle trajectories are unique and $C^1$ in time for almost every choice of initial condition in $\Omega$. This degree of regularity is more than can currently be guaranteed for weak solutions ($u \in L^1(0, T; L^\infty)$) but significantly less than that known to ensure that $u$ is regular ($u \in L^2(0, T; L^\infty)$). We rely heavily on partial regularity results due to Caffarelli, Kohn, & Nirenberg and Ladyzhenskaya & Seregin.

1. Introduction

In this paper we consider the flow of an incompressible fluid in a bounded three-dimensional domain $\Omega$ (with $C^2$ boundary) governed by the unforced Navier-Stokes equations

\begin{align*}
  u_t - v \Delta u + (u \cdot \nabla)u + \nabla p &= 0 \quad \text{(1)} \\
  \nabla \cdot u &= 0 \quad \text{(2)}
\end{align*}

with Dirichlet boundary condition $u|_{\partial\Omega} = 0$ and initial condition

\begin{equation}
  u(x, 0) = u_0. \quad \text{(3)}
\end{equation}
The coefficient $\nu > 0$ is the kinematic viscosity of the fluid, $u$ denotes the velocity, and $p$ is the pressure. It is still an open question whether or not the system (1)–(3) can develop a singularity, even if $u_0$ is smooth. Nevertheless, the existence of a weak solution $u$ of the system (1)–(3) has been known for more than a half century due to the work of Leray [17] and Hopf [13]: recall that $u \in L^\infty(0, T; H) \cap L^2(0, T; H_0^1(\Omega))$ is a weak solution of (1)–(3) if it satisfies (1) in a distributional sense. ($H$ denotes the closure of divergence-free smooth functions with compact support in $\Omega$ with respect to the norm of $[L^2(\Omega)]^3$.)

It is not known whether weak solutions have any physically desirable properties: there is currently no proof that they are unique, and while it has been proved that there exist weak solutions satisfying the energy inequality,

$$\|u(t)\|^2 + \int_0^t \|Du(s)\|^2 \, ds \leq \|u_0\|^2,$$

we do not know if this property is enjoyed by every weak solution.

However, each weak solution is sufficiently regular to define the Lagrangian trajectories of ‘fluid particles’, but again uniqueness of these trajectories is an open problem. More precisely, for a given weak solution $u$ with $u_0 \in H_0^1 \cap H$, Foias, Guillopé, & Temam [11] showed that for every $a \in \Omega$ there exists a continuous function $\xi : [0, T] \to \Omega$ satisfying

$$\xi(t) = a + \int_0^t u(\xi(s), s) \, ds,$$

i.e. the integral form of the ODE $\dot{\xi} = u(\xi(t), t)$ with $\dot{\xi}(0) = a$. However, the solution of (4) may not be unique. Thus one cannot exclude a priori that a single weak solution $u$ may give rise to many completely different flows of fluid particles, and it is not obvious that one can choose a collection of solutions of (4) that fit together in a consistent way.

However, Foias et al. show that given a weak solution $u$ there does exist at least one ‘solution map’ $\Phi : \Omega \times [0, T] \to \Omega$ such that

(i) $\xi_a(\cdot) = \Phi(a, \cdot)$ satisfies (4),
(ii) $\xi_a(\cdot) \in W^{1,1}(0, T)$,
(iii) the mapping $a \mapsto \Phi(a, \cdot)$ belongs to $L^\infty(\Omega; C([0, T], \tilde{\Omega}))$, and
(iv) $\Phi$ is volume-preserving, in the sense that for all continuous functions $g$ with compact support in $\Omega$,

$$\int_\Omega g(\Phi(x, t)) \, dx = \int_\Omega g(x) \, dx.$$  

Note that, extending (5) by continuity to hold for all Borel bounded functions on $\Omega$ and then setting $g$ to be the characteristic function of $B$, (iv) implies that

$$\mu[\Phi(\cdot, t)^{-1}(B)] = \mu(B),$$

i.e. $\Phi(\cdot, t)$ is volume-preserving in a more conventional sense.

Subsequent work by DiPerna & Lions [9], based on a generalised notion of a solution of (4) in terms of a solution of the transport equation

$$\rho_t + (u \cdot \nabla)\rho = 0,$$