Quantum Variance of Maass-Hecke Cusp Forms

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Abstract: In this paper we study quantum variance for the modular surface $X = \Gamma \backslash \mathbb{H}$, where $\Gamma = \text{SL}_2(\mathbb{Z})$ is the full modular group. We evaluate asymptotically the quantum variance, which is introduced by S. Zelditch and describes the fluctuations of a quantum observable. It is shown that the quantum variance is equal to the classical variance of the geodesic flow on $S^*X$, the unit cotangent bundle of $X$, but twisted by the central value of the Maass-Hecke $L$-functions.

1. Introduction

Let $X = \Gamma \backslash \mathbb{H}$ be the arithmetic modular surface, where $\Gamma = \text{SL}(2, \mathbb{Z})$ and $\mathbb{H}$ the upper half plane. On $X$ we have the normalized invariant hyperbolic measure $d\mu = \frac{3}{\pi} \frac{dxdy}{y^2}$, and the Beltrami-Laplace operator

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

In addition to the Laplace operator, we have the commuting family of Hecke operators $T_n, n \geq 1$; and it is well known that all $T_n$’s commute with $\Delta$. For the Maass-Hecke eigenforms, i.e., the Maass form $\phi$ satisfies

$$\Delta \phi + \lambda \phi = 0, \quad T_n \phi = \lambda_n \phi,$$

with Laplacian eigenvalue $\frac{1}{4} + t^2$ and has a Fourier expansion of the type

$$\phi(z) = \sqrt{y} \sum_{n \neq 0} \rho_\phi(n) K_{it}(2\pi |n|y)e(nx),$$

where $K_{it}$ is the $K$-Bessel function and $\rho_\phi(n)$ is proportional to the $n^{th}$ Hecke eigenvalue $\lambda_\phi(n)$, i.e., $\rho_\phi(n) = \lambda_\phi(n) \rho_\phi(1)$. 
We are mainly interested in the distribution of the probability measures on $X$, $d\mu_j := |\phi_j(z)|^2 d\mu$, as $j \to \infty$.

The quantum unique ergodicity conjecture (QUE) for $X$ is a deep problem concerning the mass equidistribution of automorphic forms on the modular surface $X$. Its goal is to understand the limiting behavior of Laplacian eigenfunctions. This is an important problem in mathematical physics and number theory and has its origin from Quantum Chaos, which attempts to understand the relation between classical physics and quantum physics. In the number theory setting, we consider the case of Maass cusp forms associated with large Laplace eigenvalues for $X$.

Schnirelman [30], Colin de Verdiere [4] and Zelditch [36] have shown in general that if the geodesic flow on the unit cotangent bundle of a manifold $Y$ is ergodic, (by a similar definition for $X$, we form $d\mu_j$ and $d\mu$ on $Y$) there exists a full density subsequence $\{\phi_{jk}\}$ of $\{\phi_j\}$ which becomes equidistributed as $j_k \to \infty$, i.e.

$$\lim_{j_k \to \infty} \int_Y \psi d\mu_{j_k} = \int_Y \psi d\mu.$$ 

for any Schwartz function $\psi \in \mathcal{C}_0^{\infty}(Y)$. The geodesic flow being ergodic means almost all orbits of the flow become equidistributed. The above result can be viewed as the quantum analogue of the geodesic flow being geodesic. Works of Hejhal-Rackner [12] and Rudnick-Sarnak [27] suggest that there are no exceptional subsequences such that $d\mu_j \to d\mu$ as $j \to \infty$, that is called quantum unique ergodicity. More precisely, the quantum unique ergodicity conjecture for $X$, recently established by the works of Lindenstrauss and Soundararajan, states that:

**Conjecture 1.** For any Jordan measurable compact region $A \subset X$, we have

$$\lim_{j \to \infty} \int_A d\mu_j = \int_A d\mu.$$

This is equivalent to

$$\lim_{j \to \infty} \int_X \psi d\mu_j = \int_X \psi d\mu$$

for any Schwartz function $\psi \in \mathcal{C}_0^{\infty}(X)$.

From Watson’s explicit triple product formula [34], let $f$ be a Maass-Hecke eigenform,

$$|\int_X f(z) d\mu_j|^2 = \frac{\Lambda \left( \frac{1}{2}, f \otimes \text{sym}^2 \phi_j \right) \Lambda \left( \frac{1}{2}, f \right)}{\Lambda(1, \text{sym}^2 f) \Lambda(1, \text{sym}^2 \phi_j)^2},$$

where $\Lambda$ is the completed $L$-function with the infinite part included. The special values $L(1, \text{sym}^2 \cdot)$ have not much effect since we have the following effective bounds due to Iwaniec and Hoffstein-Lockhart, for any $\epsilon > 0$:

$$\lambda_{j}^{-\epsilon} \ll_{\epsilon} L(1, \text{sym}^2 \phi_j) \ll_{\epsilon} \lambda_{j}^{\epsilon}.$$

Thus, the subconvexity bound of the degree 6 triple product $L$-function

$$L \left( \frac{1}{2}, f \otimes \text{sym}^2 \phi_j \right) = o \left( \lambda_{j}^{\frac{1}{2}} \right)$$