There is No “Theory of Everything” Inside E8

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Abstract: We analyze certain subgroups of real and complex forms of the Lie group E₈, and deduce that any “Theory of Everything” obtained by embedding the gauge groups of gravity and the Standard Model into a real or complex form of E₈ lacks certain representation-theoretic properties required by physical reality. The arguments themselves amount to representation theory of Lie algebras in the spirit of Dynkin’s classic papers and are written for mathematicians.

1. Introduction

Recently, the preprint [1] by Garrett Lisi has generated a lot of popular interest. It boldly claims to be a sketch of a “Theory of Everything”, based on the idea of combining the local Lorentz group and the gauge group of the Standard Model in a real form of E₈ (necessarily not the compact form, because it contains a group isogenous to SL₂(C)). The purpose of this paper is to explain some reasons why an entire class of such models—which include the model in [1]—cannot work, using mostly mathematics with relatively little input from physics.

The mathematical set up is as follows. Fix a real Lie group E. We are interested in subgroups SL₂(C) and G of E so that:

\[ G \text{ is connected, compact, and centralizes } \text{SL}(2, \mathbb{C}). \] (ToE1)

We complexify and then decompose Lie(E) ⊗ ⊕C as a direct sum of representations of SL(2, C) and G. We identify SL(2, C) ⊗R ⊕C with SL₂,C × SL₂,C and write

\[ \text{Lie}(E) = \bigoplus_{m,n \geq 1} m \otimes n \otimes V_{m,n}, \] (1.1)

where m and n denote the irreducible representation of SL₂,C of that dimension and V_{m,n} is a complex representation of G ⊗R ⊕C. (Physicists would usually write 2 and 2̄ instead of 2 ⊗ 1 and 1 ⊗ 2.) Of course,
\[ m \otimes n \otimes V_{m,n} \simeq n \otimes m \otimes V_{m,n}, \]

and since the action of \( SL(2, \mathbb{C}) \cdot G \) on \( \text{Lie}(E) \) is defined over \( \mathbb{R} \), we deduce that \( V_{m,n} \simeq V_{n,m} \). We further demand that

\[ V_{m,n} = 0 \text{ if } m + n > 4, \quad \text{and} \quad \text{(ToE2)} \]

\[ V_{2,1} \text{ is a complex representation of } G. \quad \text{(ToE3)} \]

We recall the definition of complex representation and explain the physical motivation for these hypotheses in the next section. Roughly speaking, (ToE1) is a trivial requirement based on trying to construct a Theory of Everything along the lines suggested by Lisi, (ToE2) is the requirement that the model not contain any “exotic” higher-spin particles, and (ToE3) is the statement that the gauge theory (with gauge group \( G \)) is chiral, as required by the Standard Model. In fact, physics requires slightly stronger hypotheses on \( V_{m,n} \), for \( m + n = 4 \). We will not impose the stronger version of (ToE2).

**Definition 1.1.** A candidate ToE subgroup of a real Lie group \( E \) is a subgroup generated by a copy of \( SL(2, \mathbb{C}) \) and a subgroup \( G \) such that (ToE1) and (ToE2) hold. A ToE subgroup is a candidate ToE subgroup for which (ToE3) also holds.

Our main result is:

**Theorem 1.2.** There are no ToE subgroups in (the transfer of) the complex \( E_8 \) nor in any real form of \( E_8 \).

**Notation.** Unadorned Lie algebras and Lie groups mean ones over the real numbers. We use a subscript \( \mathbb{C} \) to denote complex Lie groups—e.g., \( SL_2, \mathbb{C} \) is the (complex) group of 2-by-2 complex matrices with determinant 1. We can view a \( d \)-dimensional complex Lie group \( G_{\mathbb{C}} \) as a \( 2d \)-dimensional real Lie group, which we denote by \( R(G_{\mathbb{C}}) \). (Algebraists call this operation the “transfer” or “Weil restriction of scalars”; geometers, and many physicists, call this operation “realification.”) We use the popular notation of \( SL(2, \mathbb{C}) \) for the transfer \( R(SL_2, \mathbb{C}) \) of \( SL_2, \mathbb{C} \); it is a double covering of the “restricted Lorentz group”, i.e., of the identity component \( SO(3, 1)_0 \) of \( SO(3, 1) \).

**Strategy and main results.** Our strategy for proving Theorem 1.2 will be as follows. We will first catalogue, up to conjugation, all possible embeddings of \( SL(2, \mathbb{C}) \) satisfying the hypotheses of (ToE2). The list is remarkably short. Specifically, for every candidate ToE subgroup of \( E \), the group \( G \) is contained in the maximal compact, connected subgroup \( G_{\text{max}} \) of the centralizer of \( SL(2, \mathbb{C}) \) in \( E \). The proof of Theorem 1.2 shows that the only possibilities are:

<table>
<thead>
<tr>
<th>( E )</th>
<th>( G_{\text{max}} )</th>
<th>( V_{2,1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_8(-24) )</td>
<td>( \text{Spin}(11) )</td>
<td>32</td>
</tr>
<tr>
<td>( E_8(8) )</td>
<td>( \text{Spin}(5) \times \text{Spin}(7) )</td>
<td>(4, 8)</td>
</tr>
<tr>
<td>( E_8(-24) )</td>
<td>( \text{Spin}(9) \times \text{Spin}(3) )</td>
<td>(16, 2)</td>
</tr>
<tr>
<td>( R(E_8, \mathbb{C}) )</td>
<td>( E_7 )</td>
<td>56</td>
</tr>
<tr>
<td>( R(E_8, \mathbb{C}) )</td>
<td>( \text{Spin}(12) )</td>
<td>32 ( \oplus ) 32'</td>
</tr>
<tr>
<td>( R(E_8, \mathbb{C}) )</td>
<td>( \text{Spin}(13) )</td>
<td>64</td>
</tr>
</tbody>
</table>

We then note that the representation \( V_{2,1} \) of \( G_{\text{max}} \) (and hence, of any \( G \subseteq G_{\text{max}} \)) has a self-conjugate structure. In other words, (ToE3) fails.