Dynamical Collapse of Boson Stars

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Abstract: We study the time evolution in a system of $N$ bosons with a relativistic dispersion law interacting through a Newtonian gravitational potential with coupling constant $G$. We consider the mean field scaling where $N$ tends to infinity, $G$ tends to zero and $\lambda = GN$ remains fixed. We investigate the relation between the many body quantum dynamics governed by the Schrödinger equation and the effective evolution described by a (semi-relativistic) Hartree equation. In particular, we are interested in the super-critical regime of large $\lambda$ [the sub-critical case has been studied in Elgart and Schlein (Comm Pure Appl Math 60(4):500–545, 2007) and Knowles and Pickl (Commun Math Phys 298(1):101–138, 2010)], where the nonlinear Hartree equation is known to have solutions which blow up in finite time. To inspect this regime, we need to regularize the interaction in the many body Hamiltonian with an $N$ dependent cutoff that vanishes in the limit $N \to \infty$. We show, first, that if the solution of the nonlinear equation does not blow up in the time interval $[-T, T]$, then the many body Schrödinger dynamics (on the level of the reduced density matrices) can be approximated by the nonlinear Hartree dynamics, just as in the sub-critical regime. Moreover, we prove that if the solution of the nonlinear Hartree equation blows up at time $T$ (in the sense that the $H^{1/2}$ norm of the solution diverges as time approaches $T$), then also the solution of the linear Schrödinger equation collapses (in the sense that the kinetic energy per particle diverges) if $t \to T$ and, simultaneously, $N \to \infty$ sufficiently fast. This gives the first dynamical description of the phenomenon of gravitational collapse as observed directly on the many body level.

1. Introduction and Main Results

We consider systems of gravitating bosons known as boson stars. Assuming the particles to have a relativistic dispersion, but the interaction to be treated classically (Newtonian
gravity), we arrive at the $N$-particle Hamiltonian

$$H_{\text{grav}} = \sum_{j=1}^{N} \sqrt{1 - \Delta x_j} - G \sum_{i<j}^{N} \frac{1}{|x_i - x_j|}$$

acting on the Hilbert space $L^2_s(\mathbb{R}^{3N})$, the subspace of $L^2(\mathbb{R}^{3N})$ containing all functions symmetric with respect to arbitrary permutations (here we use units with $\hbar = 1$, $c = 1$, and $m = 1$, where $m$ denotes the mass of the bosons).

We are interested in the mean field limit where $N \to \infty$, $G \to 0$ so that $NG =: \lambda$ remains fixed. In other words, we are going to study a family of systems, parametrized by the number of bosons $N$, described by the $N$-particle Hamiltonian

$$H_N = \sum_{j=1}^{N} \sqrt{1 - \Delta x_j} - \frac{\lambda}{N} \sum_{i<j}^{N} \frac{1}{|x_i - x_j|}. \quad (1.1)$$

The system is critical, and it behaves very differently depending on the value of the coupling constant $\lambda > 0$. The criticality of the system is a consequence of the fact that the kinetic energy scales, for large momenta, like the potential energy (both scales as an inverse length). The potential energy can be made arbitrarily large (and negative) by moving the particles closer and closer together ($N$ particle in a box of volume $\ell^3$ have a potential energy of the order $N \ell^{-1}$, taking also into account the $1/N$ factor in front of the interaction energy). However, in order to localize particles in a small volume we have to pay a price in terms of kinetic energy (to localize $N$ particles within a box of volume $\ell^3$, we need an energy proportional to $N \ell^{-1}$). This simple observation implies that, for small values of the coupling constant $\lambda$, the kinetic energy dominates the potential energy, and that, for sufficiently large $\lambda$, the kinetic energy needed to bring particles together is not sufficient to compensate for the gain in the potential energy.

For every $N \in \mathbb{N}$, there exists therefore a critical coupling constant $\lambda_{\text{crit}}(N)$ such that $H_N$ is bounded below for all $\lambda < \lambda_{\text{crit}}(N)$ and such that

$$\inf_{\psi \in L^2(\mathbb{R}^{3N})} \frac{\langle \psi, H_N \psi \rangle}{\|\psi\|^2} = -\infty$$

for all $\lambda > \lambda_{\text{crit}}(N)$. It was proven in [17] that the critical constant is given, as $N \to \infty$, by the critical coupling constant for the Hartree energy functional

$$\mathcal{E}_{\text{Hartree}}(\varphi) = \int dx \left| (1 - \Delta)^{1/4} \varphi(x) \right|^2 - \frac{\lambda}{2} \int dx dy \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x - y|}. \quad (1.2)$$

More precisely, it was proven in [17] that, as $N \to \infty$, $\lambda_{\text{crit}}(N) \to \lambda_{\text{crit}}^H$, where

$$\frac{1}{\lambda_{\text{crit}}^H} = \sup_{\varphi \in L^2(\mathbb{R}^3), \|\varphi\|=1} \frac{1}{2} \frac{\int dx dy |\varphi(x)|^2 |\varphi(y)|^2 |x - y|^{-1}}{\int dx \|\nabla\|^{1/2} |\varphi(x)|^2}.$$

Note that, with this definition, $\mathcal{E}_{\text{Hartree}}(\varphi) \geq 0$ for all $\varphi \in H^{1/2}(\mathbb{R}^3)$ if $\lambda \leq \lambda_{\text{crit}}^H$, while

$$\inf_{\varphi \in H^{1/2}(\mathbb{R}^3), \|\varphi\|=1} \mathcal{E}_{\text{Hartree}}(\varphi) = -\infty.$$