Global Well–Posedness of the 3D Primitive Equations with Partial Vertical Turbulence Mixing Heat Diffusion

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Abstract: The three–dimensional incompressible viscous Boussinesq equations, under the assumption of hydrostatic balance, govern the large scale dynamics of atmospheric and oceanic motion, and are commonly called the primitive equations. To overcome the turbulence mixing a partial vertical diffusion is usually added to the temperature advection (or density stratification) equation. In this paper we prove the global regularity of strong solutions to this model in a three-dimensional infinite horizontal channel, subject to periodic boundary conditions in the horizontal directions, and with no-penetration and stress-free boundary conditions on the solid, top and bottom boundaries. Specifically, we show that short time strong solutions to the above problem exist globally in time, and that they depend continuously on the initial data.

1. Introduction

The partial differential equation model that describes convective flow in ocean dynamics is known to be the Boussinesq equations, which are the Navier–Stokes equations (NSE) of incompressible flows with rotation coupled to the heat (or density stratification) and salinity transport equations. The questions of the global well–posedness of the 3D Navier–Stokes equations are considered to be among the most challenging mathematical problems. In the context of the atmosphere and the ocean circulation dynamics geophysicists take advantage of the shallowness of the oceans and the atmosphere to simplify the Boussinesq equations by modeling the vertical motion with the hydrostatic balance. This leads to the well-known primitive equations for ocean and atmosphere dynamics (see, e.g., [24,25,29,31,32,37,38] and references therein). A vertical heat diffusivity is usually added as a leading order approximation to the effect of micro-scale turbulence mixing (cf., e.g., [16,17,24]). As a result one arrives to the following dimensionless 3D variant of the primitive equations (Boussinesq equations):

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla H)v + w\frac{\partial v}{\partial z} + f_0 \vec{k} \times v + \nabla H p + L_1 v = 0,
\]  
(1)
\[ \frac{\partial z p + T}{T} = 0, \]  
\[ \nabla_H \cdot v + \partial_z w = 0, \]  
\[ \frac{\partial T}{\partial t} + v \cdot \nabla_H T + w \frac{\partial T}{\partial z} + L_2 T = Q, \]  

where the horizontal velocity vector field \( v = (v_1, v_2) \), the velocity vector field \((v_1, v_2, w)\), the temperature \( T \) and the pressure \( p \) are the unknowns. \( f_0 \) is the Coriolis parameter, \( Q \) is a given heat source. For simplicity, we drop the coupling with the salinity equation, which is an advection diffusion equation, but the results reported here will be equally valid with the addition of the coupling with the salinity. Moreover, we also assume for simplicity that \( Q \) is time independent. The viscosity and the heat vertical diffusion operators \( L_1 \) and \( L_2 \), respectively, are given by

\[ L_1 = -\frac{1}{R_1} \Delta_H - \frac{1}{R_2} \frac{\partial^2}{\partial z^2}, \]
\[ L_2 = -\frac{1}{R_3} \frac{\partial^2}{\partial z^2}, \]

where \( R_1, R_2 \) are positive constants representing the horizontal and vertical dimensionless Reynolds numbers, respectively, and \( R_3 \) is a positive constant which stands for the vertical dimensionless eddy heat diffusivity turbulence mixing coefficient (cf., e.g., [16, 17]). We set \( \nabla_H = (\partial_x, \partial_y) \) to be the horizontal gradient operator and \( \Delta_H = \partial_x^2 + \partial_y^2 \) to be the horizontal Laplacian. We denote by

\[ \Gamma_u = \{(x, y, 0) \in \mathbb{R}^3\}, \]
\[ \Gamma_b = \{(x, y, -h) \in \mathbb{R}^3\}, \]

the upper and lower solid boundaries, respectively. We equip system (1)–(4), on the physical top and bottom boundaries, with the following no–normal flow and stress free boundary conditions for the flow velocity vector field \((v, w)\), namely,

\[ \text{on } \Gamma_u : \frac{\partial v}{\partial z} = 0, \ w = 0, \]  
\[ \text{on } \Gamma_b : \frac{\partial v}{\partial z} = 0, \ w = 0, \]

and for simplicity, we set the Dirichlet boundary condition for \( T \):

\[ T|_{z=0} = 0, \quad T|_{z=-h} = 1. \]

Horizontally, we set \((v, w)\) and \( T \) to satisfy periodic boundary conditions:

\[ v(x + 1, y, z) = v(x, y + 1, z) = v(x, y, z); \]  
\[ w(x + 1, y, z) = w(x, y + 1, z) = w(x, y, z); \]  
\[ T(x + 1, y, z) = T(x, y + 1, z) = T(x, y, z). \]

We will denote by

\[ M = (0, 1)^2 \quad \text{and} \quad \Omega = M \times (-h, 0). \]