A Normal Form for the Schrödinger Equation with Analytic Non-linearities

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Abstract: We discuss a class of normal forms of the completely resonant non-linear Schrödinger equation on a torus. We stress the geometric and combinatorial constructions arising from this study.

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1. Introduction

In this paper we exhibit a normal form, with remarkable integrability properties, for the completely resonant non-linear Schrödinger equation on the torus $\mathbb{T}^n$, $n \in \mathbb{N}$ (NLS for brevity):

$$-i u_t + \Delta u = \kappa |u|^{2q} u + \partial_{\bar{u}} G(|u|^2), \quad q \geq 1 \in \mathbb{N}. \quad (1)$$

where $u := u(t, \varphi), \varphi \in \mathbb{T}^n$ and $G(\alpha)$ is a real analytic function whose Taylor series starts from degree $q + 2$. The case $q = 1$ is of particular interest and is usually referred to as the cubic NLS.

It is well known that Eq. 1, the NLS, can be written as an infinite dimensional Hamiltonian dynamical system.

It has the energy $H = \int_{\mathbb{T}^n} (|\nabla (u)|^2 + \kappa (q + 1)^{-1} |u|^{2(q+1)} + G(|u|^2)) \frac{d\phi}{(2\pi)^n}$, the moment $M = \int_{\mathbb{T}^n} \bar{u}(\varphi) \nabla u(\varphi) \frac{d\phi}{(2\pi)^n}$ and the mass $L = \int_{\mathbb{T}^n} |u(\varphi)|^2 \frac{d\phi}{(2\pi)^n}$, as integrals of motion.

Passing to the Fourier representation

$$u(t, \varphi) := \sum_{k \in \mathbb{Z}^n} u_k(t)e^{i(k, \varphi)}, \quad (2)$$

we have, up to a rescaling of $u$ and of time, in coordinates:

$$H := \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k \pm \sum_{k_1, k_2, k_3 \in \mathbb{Z}^n} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4} \cdots u_{k_{2q+1}} \bar{u}_{k_{2q+2}} + \int_{\mathbb{T}^n} G(|u|^2) \frac{d\phi}{(2\pi)^n}. \quad (3)$$

We fix the sign to be $+$ since in our treatment it does not play any particular role.