On Ternary Quotients of Cubic Hecke Algebras

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Abstract: We prove that the quotient of the group algebra of the braid group introduced by Funar (Commun Math Phys 173:513–558, 1995) collapses in characteristic distinct from 2. In characteristic 2 we define several quotients of it, which are connected to the classical Hecke and Birman-Wenzl-Murakami quotients, but which admit in addition a symmetry of order 3. We also establish conditions on the possible Markov traces factorizing through it.

1. Introduction

Let $B_n$ be the braid group on $n$ strings ($n \geq 2$), that is the group defined by $n - 1$ generators $s_1, \ldots, s_{n-1}$ submitted to the relations $s_is_j = s_js_i$ whenever $i - j \geq 2$, and $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for any $i = 1, \ldots, n - 2$ (see e.g. [Bi] or [KM] for basic results on these groups).

This paper grew out of an attempt to understand the mysterious ‘cubic Hecke algebras’ defined by L. Funar and used in [F1] and [BF]. In [F1], an algebra $K_n(\gamma)$ for $\gamma \in k$ is defined over a commutative ring $k$ as the quotient of the group algebra $kB_n$ of the braid group $B_n$ on $n$ strands, by the relations $s_i^3 = \gamma$, and $s_is_{i+1}s_is_{i+1} + s_is_i^2s_{i+1} + s_is_{i+1}s_is_{i+1} + s_is_{i+1}s_{i+1}s_i + s_is_{i+1}s_i + s_i^2s_{i+1} + s_i^2s_{i+1}s_i + \gamma s_i + \gamma s_{i+1} = 0$. Notice that the relations are equivalent to $s_i^3 = \gamma$, $s_is_2^2s_1s_2 + s_1s_2^2s_1 + s_1^2s_2s_1 + s_1s_2s_1^2 + s_1^2s_2^2 + s_2^2s_1^2 + \gamma s_1 + \gamma s_2 = 0$. The striking property of this algebra is that the latter relation involves only $s_1$, $s_2$ and that, as proved in [F1], it is a finitely generated $k$-module (hence finite dimensional over $k$ if $k$ is a field).

Although many finite-dimensional cubic quotients of the (group algebra of the) braid groups have been defined, to our knowledge it is the only one which is not a quotient of the classical Birman-Wenzl-Murakami algebra and which can be defined from relations in $kB_3$. Notice that, whenever $\gamma$ admits an invertible third root $\alpha \in k$ with $\alpha^3 = \gamma$, we have $K_n(\gamma) \cong K_n(1)$ under $s_i \mapsto \alpha^{-1}s_i$ – and in particular always $K_n(-1) \cong K_n(1)$. Moreover, $K_n(1)$ is a quotient of the group algebra $k\Gamma_n$, for $\Gamma_n = B_n/ \langle s_1^3 \rangle$. This group $\Gamma_n$ is a semidirect product $\Gamma_n^0 \rtimes C_3$, with $C_k$ denoting the cyclic group of order...
k, and the defining ideal of $K_n(1)$ has the remarkable property to be generated by a $C_3$-invariant ideal in $\mathbb{Z} \Gamma_3^0$ – thus deserving the name ternary used in the title.

By a theorem of Coxeter, $\Gamma_n$ is finite if and only if $n \leq 5$. Moreover, in this case it is a finite complex reflection group, and, as was conjectured by Broué, Malle and Rouquier, $k \Gamma_n$ for $n \leq 5$ admits a flat deformation similar to the presentation of the ordinary Hecke algebra as a deformation of $k S_n$. This has been proved in [BM], Satz 4.7 for $n = 3, 4$, and recently in [M] for $n = 5$. Partly stimulated by this conjecture, the authors of [BF] constructed a deformation of $K_n(\gamma)$ (still finitely generated).

The main motivation in [F1] and [BF] is to construct link invariants. In [F1] it is claimed that $K_n(-1)$ admits a Markov trace with values in $\mathbb{Z}/6\mathbb{Z}$. A more general statement is claimed in [BF], that the constructed deformation provides a link invariant with values in some extended ring. Around 2004–2005, S. Orevkov pointed out a gap in a part of [BF] devoted to the proof of the invariance of the trace under Markov moves, which originates in [F1]. In 2008, the second author of the present paper noticed that, when $k$ is a field of characteristic 0, the ’tower of algebras’ $K_n(1)$ collapsed, more precisely that $K_n(1) = 0$ for $n \geq 5$ (see Theorem 4.8 below). However, when $k = \mathbb{Z}$, this tower does not collapse. This can be seen from the fact that the natural group morphisms $\Gamma_n \to C_3$ induce morphisms $\mathbb{Z} \Gamma_n \to \mathbb{Z} C_3 \to (\mathbb{Z}/8\mathbb{Z}) C_3$ which factorize through $K_n(1)$.

1.1. Statement of the main results. Letting $K_n = K_n(1)$ we prove (see Corollary 4.3 and Theorems 4.8 and 4.9)

Theorem. When $k = \mathbb{Z}$,

(i) $K_n$ is a finite $\mathbb{Z}$-module for $n \geq 5$.
(ii) The exponent (as an abelian group) of $K_n$ has the form $2^r 3^s$ for some $r, s$ (depending on $n$) when $n \geq 5$.
(iii) The exponent of $K_n$ is a power of 2 (not depending on $n$) when $n \geq 7$.

When $k$ is a field, in order to get a stably nontrivial structure, we thus need to assume that $k$ has characteristic 2.

Theorem. Assume $k$ is a field of characteristic 2. For all $n$, there exists a quotient $\mathcal{H}_n$ of $K_n$, which has dimension $3(n! - 1)$ and which embeds inside a product of three Hecke algebras. This algebra $\mathcal{H}_n$ is the quotient of $k \Gamma_n$ by the relation $s_1 s_2^{-1} + s_2 s_1^{-1} + s_1^{-1} s_2 + s_2^{-1} s_1 = 0$.

We call this algebra the ternary Hecke algebra, as it can be defined as the quotient of $k \Gamma_n$ by the intersection of the three ideals whose corresponding quotients define the three possible Hecke algebras at third roots of 1.

Taking $k = \mathbb{Z}$, we let $K_{\infty}$ denote the direct limit of the $K_n$ under the natural morphisms $K_n \to K_{n+1}$, and we similarly define $\mathcal{H}_{\infty}$. Using the second definition above, $\mathcal{H}_{\infty}$ can be defined over $\mathbb{Z}/4\mathbb{Z}$.

We recall that a Markov trace on $K_{\infty}$ is a $\mathbb{Z}$-module morphism $t : K_{\infty} \to M$, where $M$ is some $\mathbb{Z}[u, v]$-module, which satisfies $t(xy) = t(yx)$ for all $x, y \in K_{\infty}$, $t(x s_n) = u t(x)$ and $t(x s_n^{-1}) = v t(x)$ for all $x$ in (the image of) $K_n$. It can be shown that such a Markov trace, if it exists, is uniquely determined by the value $t(1)$, and takes values in $\mathbb{Z}[u, v]t(1) \subset M$.

Theorem. (i) If $t : K_{\infty} \to \mathbb{Z}[u, v]t(1)$ is a Markov trace, then $16 t(1) = 0$, $4uv : t(1) = 4t(1)$, $4u^3 : t(1) = 4v^3 : t(1) = -4t(1)$ and $(3u^3 + 3v^3 - 3uv + 1)t(1) = 0$. 