Multi-scale Analysis of Compressible Viscous and Rotating Fluids

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Abstract: We study a singular limit for the compressible Navier-Stokes system when the Mach and Rossby numbers are proportional to certain powers of a small parameter $\varepsilon$. If the Rossby number dominates the Mach number, the limit problem is represented by the 2-D incompressible Navier-Stokes system describing the horizontal motion of vertical averages of the velocity field. If they are of the same order then the limit problem turns out to be a linear, 2-D equation with a unique radially symmetric solution. The effect of the centrifugal force is taken into account.

1. Introduction

Rotating fluid systems appear in many applications of fluid mechanics, in particular in models of atmospheric and geophysical flows, see the monograph [3]. Earth’s rotation, together with the influence of gravity and the fact that atmospheric Mach number is typically very small, give rise to a large variety of singular limit problems, where some of these characteristic numbers become large or tend to zero, see Klein [13]. We consider a simple situation, where the Rossby number is proportional to a small parameter $\varepsilon$, while the Mach number behaves like $\varepsilon^m$, with $m \geq 1$. Scaling of this type with various choices of $m$ appears, for instance, in meteorological models (cf. [13, Sect. 1.3]).

We neglect the influence of the temperature and we write the equations of motion in the rotating frame attached to the Earth. Assuming that the rotation axis is parallel to $x_3$, we set $\mathbf{b} = [0, 0, 1]$, and the associated centrifugal force is denoted $\nabla_x G \approx \nabla_x |x_3|^2$, where we have written $x_3 = [x_1, x_2]$, so that the time derivative of the momentum field $\varrho \mathbf{u} = \varrho \mathbf{u}(t, x)$ is changed into $\partial_t (\varrho \mathbf{u}) + \frac{1}{\varepsilon} (\mathbf{b} \times \varrho \mathbf{u}) - \frac{1}{\varepsilon} \varrho \nabla_x G$. Finally we arrive

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at the following scaled Navier-Stokes system describing the time evolution of the fluid density $\varrho = \varrho(t, x)$ and the velocity field $\mathbf{u} = \mathbf{u}(t, x)$:

$$
\partial_t \varrho + \text{div}_x (\varrho \mathbf{u}) = 0, \quad (1.1)
$$

$$
\partial_t (\varrho \mathbf{u}) + \text{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon} (\mathbf{b} \times \varrho \mathbf{u}) + \frac{1}{\varepsilon^2 m} \nabla_x \varrho = \text{div}_x \mathbf{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x \mathbf{G}, \quad (1.2)
$$

where $p$ is the pressure, and $\mathbf{S}$ is the viscous stress tensor determined by Newton’s rheological law

$$
\mathbf{S}(\nabla_x \mathbf{u}) = \mu \left( \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \text{div}_x \mathbf{u} \right), \quad \mu > 0. \quad (1.3)
$$

For the sake of simplicity, we have omitted possible influence of the so-called bulk viscosity component in the viscous stress.

We consider a very simple geometry of the underlying physical space $\Omega \subset \mathbb{R}^3$, namely $\Omega$ is an infinite slab,

$$
\Omega = \mathbb{R}^2 \times (0, 1).
$$

Moreover, to eliminate entirely the effect of the boundary on the motion, we prescribe the complete slip boundary conditions for the velocity field:

$$
\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0, \quad [\mathbf{S} \mathbf{n}] \times \mathbf{n}|_{\partial \Omega} = 0, \quad (1.4)
$$

where $\mathbf{n}$ denotes the outer normal vector to the boundary. Note that the more common no-slip boundary condition

$$
\mathbf{u}|_{\partial \Omega} = 0
$$

would yield a trivial result in the asymptotic limit, namely $\mathbf{u} \to 0$ for $\varepsilon \to 0$. On the other hand, the so-called Navier’s boundary condition

$$
\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0, \quad \beta \mathbf{u}_{\text{tan}} + [\mathbf{S} \mathbf{n}]_{\text{tan}}|_{\partial \Omega} = 0, \quad \beta > 0, \quad (1.5)
$$

gives rise to a friction term in the limit system known as Ekman’s pumping, see Sect. 2.5 below.

As is well known (see Ebin [7]), the boundary conditions (1.4) may be conveniently reformulated in terms of geometrical restrictions imposed on the state variables that are periodic with respect to the vertical variable $x_3$. More specifically, we take

$$
\Omega = \mathbb{R}^2 \times T^1, \quad (1.6)
$$

where $T^1 = [-1, 1]|_{[-1,1]}$ is a one-dimensional torus, on which the fluid density $\varrho$ as well as the horizontal component of the velocity $\mathbf{u}_h = [u^1, u^2]$ are extended to be even in $x_3$, while the vertical component $u^3$ is taken odd:

$$
\varrho(x_1, x_2, -x_3) = \varrho(x_1, x_2, x_3), \quad u^i(x_1, x_2, -x_3) = u^i(x_1, x_2, x_3), \quad i = 1, 2, \quad (1.7)
$$

$$
u^3(x_1, x_2, -x_3) = -u^3(x_1, x_2, x_3).
$$

As shown in [3], incompressible rotating fluids stabilize to a 2D motion described by the vertical averages of the velocity provided the Rossby number $\varepsilon$ is small enough. Besides, the stabilizing effect of rotation has been exploited by many authors, see e.g.