Dislocations in an Anisotropic Swift-Hohenberg Equation

Mariana Haragus1, Arnd Scheel2

1 Université de Franche-Comté, Laboratoire de Mathématiques, 16 route de Gray, 25030 Besançon Cedex, France. E-mail: mharagus@univ-fcomte.fr
2 University of Minnesota, School of Mathematics, 206 Church St. S.E., Minneapolis, MN 55455, USA. E-mail: scheel@umn.edu

Received: 9 April 2010 / Accepted: 26 February 2012
Published online: 15 September 2012 – © Springer-Verlag 2012

Abstract: We study the existence of dislocations in an anisotropic Swift-Hohenberg equation. We find dislocations as traveling or standing waves connecting roll patterns with different wavenumbers in an infinite strip. The proof is based on a bifurcation analysis. Spatial dynamics and center-manifold reduction yield a reduced, coupled-mode system of differential equations. Existence of traveling dislocations is then established by showing that this reduced system possesses robust heteroclinic orbits.

1. Introduction

Defects in patterns occur with a striking universality and regularity in a wide range of systems. The arguably best-studied experimental setup is the Rayleigh-Bénard convection experiment, where convection roll patterns with a variety of embedded defects form close to onset. Defects in crystal patterns also play a fundamental role in material science. In both scenarios, the Swift-Hohenberg equation, which in its simplest form reads

\[ u_t = -(\Delta + 1)^2 u + \mu u - u^3, \]

on \( \mathbb{R}^n \), \( n = 1, 2, 3 \), has been used as a prototypical model system that mimics phenomena qualitatively (and sometimes even quantitatively), while being analytically and numerically easily tractable; see for instance [2]. The crucial property of the Swift-Hohenberg equation is the instability of a spatially homogeneous state with respect to roll patterns \( u(X) \sim \cos(K \cdot X) \), with \( X \in \mathbb{R}^n \) and wavenumber \( K \in \mathbb{R}^n, |K| \sim 1 \), for parameter values \( \mu \) above criticality. Depending on the nonlinearity, this leads to the creation of stable nonlinear patterns that resemble these linear modes. We are interested in situations where nonlinear roll patterns bifurcate in a stable fashion. Our aim is to prove the existence of dislocations in planar spatially extended systems. In this context, dislocations are stationary solutions in an appropriate frame of reference, that resemble roll patterns \( \cos(K \cdot X + 2\pi \varphi(X)) \) with constant orientation \( K \sim (0, 1)^T \in \mathbb{R}^2 \) and
appropriate phase shift function $\varphi(X)$ in the far field. Associated with such patterns is a topological charge, obtained by integrating the phase $\varphi$ along a circle in the far field. For dislocations, this topological charge is $\pm 1$: the number of rolls along the vertical line $(x, y) \sim (-R, y), R \gg 1$, differs from the number along the line $(x, y) \sim (+R, y)$ by one; see for instance [2,9] for some background and references.

Despite the variational structure of the problem, proving existence of such solutions appears to be a quite delicate problem. Most attempts have focused on the analysis of phase modulation equations; see for instance [3–5]. Our approach here is slightly different in spirit. We will find dislocations as solutions in a strip $(x, y) \in \mathbb{R} \times \mathbb{R}/L\mathbb{Z}$, with $L$-periodic boundary conditions in $y$; see Fig.1 for a schematic plot. Such boundary conditions accommodate roll solutions of the form $u \sim \cos(ky)$ when $k$ solves $j \cdot \frac{2\pi}{k} = L$ for some $j \in \mathbb{Z}$. When $L$ is large, this allows for wavenumbers $k_0 := j \cdot \frac{2\pi}{k} \lesssim 1 \lesssim (j + 1) \cdot \frac{2\pi}{L} =: k_+$. The corresponding roll solutions bifurcate from the trivial solutions for $\mu \sim 0$ and we can, in principle, attempt to find traveling waves (or interfaces) between these rolls with wavenumbers $k_-, k_+$. By construction, the winding number defined above is indeed $\pm 1$, when evaluated on $|x| = R, 0 \leq y \leq L$. Extended to solutions in the plane $(x, y) \in \mathbb{R} \times \mathbb{R}$, such solutions represent a periodic alignment of dislocations in the $y$-direction.

It turns out that our approach reveals an obstacle to the construction of such standing (or slow moving) dislocations in the form of resonances. In fact, together with rolls of the form $\cos(ky), k \sim 1$, we have rolls of the form $\cos(k_xx + k_yy)$, when $k_x^2 + k_y^2 \sim 1$. With the above restriction $k_y \in \frac{2\pi}{L}\mathbb{Z}$, this allows for a plethora of roll solutions compatible with the boundary conditions when $L$ is chosen large enough. Our present approach avoids this difficulty by considering an anisotropic version of the Swift-Hohenberg equation

$$\partial_t u = - (1 + \Delta)^2 u + \mu u + \beta \partial_x^2 u - u^3. \quad (1.1)$$

Here, $u$ depends upon the two spatial variables $(x, y) \in \mathbb{R}^2$ and time $t \in \mathbb{R}, \mu$ is a small real parameter, and $\beta > 0$. The effect of $\beta$ is an effective linear damping of modes with wavenumber $k_x > 0$, so that only horizontal rolls, $k_x = 0$, exist for $\mu \sim 0$.

While the damping of vertical modes effectively simplifies the problem, we believe that the resulting bifurcation problem here is related to the problem in the isotropic case. One would expect that the coupling between rolls of different orientation is weak in the isotropic case, a fact that can be made precise using reduction methods and normal...