Critical Temperature of Periodic Ising Models

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Abstract: A periodic Ising model has interactions which are invariant under translations of a full-rank sublattice \( \mathcal{L} \) of \( \mathbb{Z}^2 \). We prove an exact, quantitative characterization of the critical temperature, defined as the supremum of temperatures for which the spontaneous magnetization is strictly positive. For the ferromagnetic model, the critical temperature is the solution of a certain algebraic equation, resulting from the condition that the spectral curve of the corresponding dimer model on the Fisher graph has a real zero on the unit torus. With our technique we provide a simple proof for the exponential decay of spin-spin correlations above the critical temperature, as well as the exponential decay of the edge-edge correlations for all non-critical edge weights of the corresponding dimer model on periodic Fisher graphs.

1. Introduction

Phase transition is a central topic of statistical mechanics. A system undergoes a phase transition whenever, for some value of the temperature and other relevant thermodynamic parameters, two or more phases can coexist in equilibrium. Different properties of the pure phases manifest themselves also as discontinuities in certain observables as a function of the appropriate thermodynamic variables, e.g. discontinuity of the magnetization as a function of the magnetic field in a ferromagnet. Of special interest is the temperature at which the phase transition occurs, that is, the critical temperature.

Lebowitz and Martin-Löf [22] defined the critical temperature, \( T_c \), for Ising ferromagnets as the supremum of temperatures such that the spontaneous magnetization is strictly positive. Lebowitz [20] identified \( T_c \) with the self-dual point for an isotropic, two-dimensional square grid Ising model. Based on various correlation inequalities and differential inequalities, Aizenman, Bursky and Fernández [2] characterize the phase transition by the exponential decay of two-point spin correlations above \( T_c \), for ferromagnetic Ising models with dimension \( D \geq 2 \).
For the two-dimensional Ising model, another approach is to apply the Fisher correspondence [7], which is a measure-preserving bijection between the even spin-spin correlation functions of the Ising model on a graph $G$, and the edge probabilities of a dimer model on a decorated graph, the Fisher graph. Since then, dimer techniques have been a powerful tool in solving the two-dimensional Ising model, see for example the paper of Kasteleyn [14], and the book of McCoy and Wu [25]. However, due to the complexity related to large matrices, the two-dimensional Ising model has acquired a notorious reputation for difficulty. On the mathematical side, Kenyon [16–18] has proven numerous spectacular results about the dimer model on bipartite graphs in recent years. Those results make it possible to look at the two-dimensional Ising model from a new perspective. In our paper, we follow the dimer approach to the Ising model, and explicitly characterize the critical temperature, defined as in [22] and [2], with the zero of an algebraic equation.

**Theorem 1.1.** Let $T_c$ be the critical temperature of the ferromagnetic, two-dimensional periodic Ising model, defined by the largest temperature such that the spontaneous magnetization is strictly positive, then $0 < T_c < \infty$. $T_c$ is determined by the condition that the spectral curve of the corresponding dimer model on the Fisher graph has a real zero on $\mathbb{T}^2$.

Theorem 1.1 allows one to compute the critical temperature of arbitrary periodic Ising ferromagnets accurately, by solving an algebraic equation, see Example 6.14. The proof of Theorem 1.1 consists of 3 steps:

**Step 1.** Applying Lebowitz’s technique [21] and the FKG inequality, we prove that the weak limit of the even spin-spin correlation functions is independent of the boundary conditions as the size of the graph goes to infinity. For the uniqueness theorem about the Gibbs measure of dimer models on a more general class of non-bipartite graphs, we refer to Corollary B.1.

**Step 2.** Using an $n \times n$ torus to approximate the infinite bi-periodic graph, we express the two-point spin correlation functions as the determinant of certain block Toeplitz matrices. For basics of Toeplitz and Hankel matrices, we refer to the appendix. We prove that the determinant of the symbol of such block Toeplitz matrices is identically 1. Using the FKG inequality, we prove that there is a unique $T_{c,p}$, defined to be the lowest temperature such that the limit of the two-point spin correlation function is 0. The analytic property for the limit of the two-point spin correlation function in case the entries of the block Toeplitz matrices are analytic with respect to the reciprocal temperature follows from Widom’s theorem [28] and a result (Lemma 4.6) from operator analysis [10]. Hence $T_{c,p}$ has to satisfy the condition that the spectral curve has a real zero on $\mathbb{T}^2$.

**Step 3.** Characterization of $T_c$ and the condition that the spectral curve has a real zero on $\mathbb{T}^2$ (critical dimer weights). Since the two-point spin correlation functions are independent of different translation invariant Gibbs measures, we derive $T_{c,p} = T_c$, and at the critical temperature of the Ising model, the corresponding dimer model is also critical. We also prove that as $\beta$ (reciprocal temperature) increases from 0 to $\infty$, there is a unique $\beta_0$ ($0 < \beta_0 < \infty$), such that the spectral curve has a real zero on $\mathbb{T}^2$. The uniqueness of the critical dimer weights implies its identification with the criticality of the Ising model.

Here is a description of the structure of the paper. Section 2 proves the uniqueness of even spin-spin correlation functions for periodic ferromagnetic 2D Ising models. In Sect. 3 we discuss the correspondence between Ising models on a square grid and dimer