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Batalin-Vilkovisky Formalism in Perturbative Algebraic Quantum Field Theory

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Abstract: On the basis of a thorough discussion of the Batalin-Vilkovisky formalism for classical field theory presented in our previous publication, we construct in this paper the Batalin-Vilkovisky complex in perturbatively renormalized quantum field theory. The crucial technical ingredient is an extended notion of the renormalized time-ordered product as a binary product equivalent to the pointwise product of classical field theory. Originally, in causal perturbation theory, the time-ordered product is understood merely as a sequence of multilinear maps on the space of local functionals. Our extended notion of the renormalized time-ordered product (denoted by $\cdot_{\mathcal{T}_r}$) is consistent with the old one and we found a subspace of the quantum algebra which is closed with respect to $\cdot_{\mathcal{T}_r}$. On this space the renormalized Batalin-Vilkovisky algebra is then the classical algebra but written in terms of the time-ordered product, together with an operator which replaces the ill defined graded Laplacian of the unrenormalized theory. We identify it with the anomaly term of the anomalous Master Ward Identity of Brennecke and Dütsch. Contrary to other approaches we do not refer to the path integral formalism and do not need to use regularizations in intermediate steps.

1. Introduction

A powerful method for the treatment of quantum field theories with gauge symmetries is the Batalin-Vilkovisky formalism which extends the BRST method [11,12,14] and allows to discuss these theories without reference to a specific gauge fixing. Its main advantage is the simultaneous treatment of equations of motion and gauge symmetries in terms of homological algebra.

Its application to relevant physical theories is, however, somewhat formal, since the mathematical methods are designed for finite dimensional situations (see for example [1]) whereas the examples from physics are typically infinite dimensional. Moreover, the formulation of the so-called Quantum Master Equation (QME) which is used as the

starting point for the construction of a renormalized quantum field theory, suffers from the occurrence of ill defined terms.

The problem to incorporate the renormalization into the BV formalism is present since the first papers of Batalin and Vilkovisky [8–10]. In [10] the authors comment on this problem pointing out the existence of divergences and they propose to deal with them by applying some regularization scheme which puts the divergent terms of the QME at 0. In [60] it was proposed to use instead a regularization that gives to these terms finite non-zero values. This approach allowed to analyze the anomalies in a more systematic way and relate them to obstructions in fulfilling the QME. The regularization used in [60] is the Pauli-Villars scheme and the discussion is restricted only to the 1-loop order. A method valid for higher loop orders was proposed in [49], but the regularization scheme used there is non-local. The dimensional regularization and renormalization in the context of BV formalism were discussed in [59]. The BPHZ renormalization is discussed in [44]. All of the mentioned approaches rely on some regularization scheme and involve arbitrary choices. From the conceptual point of view it is still unclear how the QME should be interpreted in the renormalized theory. An alternative treatment of the QME which involves a certain extension of the field-antifield formalism was presented in [2].

An approach to a rigorous formulation of the QME has recently been performed by Costello [21]. He replaces the Quantum Master Equation by a family of regularized equations which are interpreted in terms of different scales. An unsatisfactory aspect of this approach (which is shared by many regularization schemes in quantum field theory) is that the problem which one wants to solve cannot be precisely formulated a priori.

Many rigorous approaches to quantum field theory are based on the euclidean version of the theory where spacetime is replaced by a Riemannian space. This makes the path integral more reliable and simplifies the analysis of singularities. Moreover, concrete calculations often give the same results, independent of the signature of the spacetime metric. But the Osterwalder-Schrader theorem [48] on which the transition between euclidean and Lorentzian structures is based holds only under certain conditions which are not generally valid for pseudo-Riemannian manifolds. Moreover, some of the crucial properties of quantum field theory, in particular the local commutativity of mutually spacelike localized observables, are not directly visible in the euclidean version. As a consequence, the fact that the dynamics within a globally hyperbolic subregion is completely independent from the dynamics outside of this region[1] has no counterpart in the euclidean theory. We therefore prefer to work directly on Lorentzian spacetimes.

The path integral can be understood as a linear functional on the space of functionals of field configurations. This functional contains in principle the information on the dynamics as well as on the state. But whereas the dynamics is locally determined the state necessarily involves global information. It is therefore desirable to disentangle these two aspects and to separate the dynamics from the specification of the state. Actually, this is the aim of Algebraic Quantum Field Theory as introduced by Haag et al. long ago [32,33], and on the basis of causal perturbation theory, as proposed by Stueckelberg [57] and Bogoliubov [13] and rigorously developed by Epstein and Glaser [28], a corresponding disentanglement is possible also for renormalized perturbative quantum field theory [16].

The basic idea is to construct inductively the time ordered product as a sequence of symmetric multilinear maps $\mathcal{T}_n$ of n local functionals of field configurations into the operator algebra of the quantum theory. This construction is (up to finite renormalizations) fixed by the requirement that

[1] For quantum field theory this was first proved in [16].