QCD on an Infinite Lattice

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Received: 11 August 2011 / Accepted: 15 October 2012
Published online: 16 February 2013 – © Springer-Verlag Berlin Heidelberg 2013

Abstract: We construct a mathematically well–defined framework for the kinematics of Hamiltonian QCD on an infinite lattice in $\mathbb{R}^3$, and it is done in a C*-algebraic context. This is based on the finite lattice model for Hamiltonian QCD developed by Kijowski, Rudolph e.a.. To extend this model to an infinite lattice, we need to take an infinite tensor product of nonunital C*-algebras, which is a nonstandard situation. We use a recent construction for such situations, developed by Grundling and Neeb. Once the field C*-algebra is constructed for the fermions and gauge bosons, we define local and global gauge transformations, and identify the Gauss law constraint. The full field algebra is the crossed product of the previous one with the local gauge transformations. The rest of the paper is concerned with enforcing the Gauss law constraint to obtain the C*-algebra of quantum observables. For this, we use the method of enforcing quantum constraints developed by Grundling and Hurst. In particular, the natural inductive limit structure of the field algebra is a central component of the analysis, and the constraint system defined by the Gauss law constraint is a system of local constraints in the sense of Grundling and Lledo. Using the techniques developed in that area, we solve the full constraint system by first solving the finite (local) systems and then combining the results appropriately. We do not consider dynamics.

1. Introduction

QCD is an important component of the standard model, and the explicit construction of a field C*-algebra for it is still an unsolved problem in mathematical physics. The construction of a field algebra is a kinematics problem and it precedes the hard problem of dynamics, which involves interactions, so it seems more tractable. There is a deep body of theory developed for the locality properties of the field algebras of quantum field theories in space-time (cf. [21] for a survey), and of course any explicitly constructed field algebra of this system must be consistent with that. There is also extensive work on the Hamiltonian model of a Fermion in a nonabelian \textit{classical} gauge potential in
Thus far, the only explicit rigorous constructions of appropriate field algebras for QCD have been for lattice approximations of Hamiltonian QCD in $\mathbb{R}^3$, cf. [27,28]. Unfortunately due to a technical problem explained below, these models have been confined to finite lattices. This is the main problem which we want to address here, i.e. we want to construct the field $C^*$-algebra for QCD on an infinite lattice in $\mathbb{R}^3$. Using this field algebra, we then want to define gauge transformations and solve the Gauss law constraint, hence identifying the physical observables.

More specifically, for the model of QCD on a finite lattice developed by Kijowski, Rudolph e.a. [27,28], one finds that the field algebra is isomorphic to the algebra of compact operators $\mathcal{K}(\mathcal{H})$ on a separable infinite dimensional Hilbert space $\mathcal{H}$. As this has (up to unitary equivalence) only one irreducible representation, one obtains a generalized von Neumann uniqueness theorem for the system. For an infinite lattice, when passing to infinitely many degrees of freedom, one has to expect inequivalent representations. Explicitly, for the gauge part of the algebra, one needs to take an infinite tensor product of the algebras associated to the links of the lattice (these are also isomorphic to $\mathcal{K}(\mathcal{H})$). This means that the standard theory for infinite tensor products does not apply. However, there is a little-known definition for an infinite tensor product of nonunital algebras developed by Blackadar cf. [3], which however has some drawbacks. Recently this approach was further developed by Grundling and Neeb in [15], where an infinite tensor product of nonunital $C^*$-algebras was constructed which has good representation properties. This is what we use for our construction of the field algebra of our model, and as expected, this new field algebra has many inequivalent representations.

Once we have the field algebra of our model, we can define (local and global) gauge transformations, extend the field algebra to include the implementers of these, and identify the Gauss law constraint. Enforcement of quantum constraints is not a simple matter, in fact compared with Quantum Electrodynamics (cf. [29,30]), the analysis of the Gauss law is much more complicated. This is due to the fact that in QCD the Gauss law constraint is neither built from gauge invariant operators nor is it linear in the gauge connection fields. Here we use the general method of enforcing quantum constraints developed by Grundling and Hurst (the $T$-procedure, cf. [16]). It is crucial for this, that the constraint system defined by the Gauss law constraint is a system of local constraints in the sense of Grundling and Lledó [20]. This allows us to solve the full constraint system by first solving the finite (local) systems and then combining the results appropriately. This method of constraint enforcement needs no gauge fixing (i.e. the selection of one representative in each gauge orbit) hence the Gribov problem does not occur.

Finally, we will discuss some of the types of physical observables which occurred in the historical papers of Kogut and Susskind [32,31] and show how to fit these into our framework. Some of these observables, e.g. Casimirs built from the colour electric fields, are unbounded, so they are not in any $C^*$-algebra. However for a finite lattice, the colour electric fields are closely related to our field $C^*$-algebra, and we show this link concretely in Subsect. 2.1. More abstractly, they generate part of the field $C^*$-algebra in the sense of Woronowicz, cf. Example 3 in Sect. 3 of [53].

In this paper, we do not consider boundary effects, and we postpone colour charge analysis to a separate project. Boundary effects were analyzed for finite lattice systems by Kijowski and Rudolph in [27], where it was shown that from the local Gauss equation one can extract a gauge invariant, additive law for operators with eigenvalues in $\mathbb{Z}_3$. As in QED, this implies a gauge invariant conservation law: the global $\mathbb{Z}_3$-valued colour