A Phase Transition for Circle Maps and Cherry Flows

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Abstract: We study $C^2$ weakly order preserving circle maps with a flat interval. The main result of the paper is about a sharp transition from degenerate geometry to bounded geometry depending on the degree of the singularities at the boundary of the flat interval. We prove that the non-wandering set has zero Hausdorff dimension in the case of degenerate geometry and it has Hausdorff dimension strictly greater than zero in the case of bounded geometry. Our results about circle maps allow to establish a sharp phase transition in the dynamics of Cherry flows.

1. Introduction

1.1. Almost smooth maps with a flat interval. We consider the class $\mathcal{L}$ of continuous circle endomorphisms $f$ of degree one for which an open arc $U$ exists such that the following properties hold:

1. The image of $U$ is one point.
2. The restriction of $f$ to $S^1 \setminus \overline{U}$ is a $C^2$-diffeomorphism onto its image.
3. Let $(a, b)$ be a preimage of $U$ under the natural projection of the real line on $S^1$. On some right-sided neighborhood of $b$, $f$ can be represented as

$$h_r \left( (x - b)^{l_r} \right),$$

where $h_r$ is a $C^2$-diffeomorphism on an open neighborhood of $b$. Analogously, there exists a $C^2$-diffeomorphism on an left-sided neighborhood of $a$ such that $f$ is of the form

$$h_l \left( (a - x)^{l_l} \right).$$

The ordered pair $(l_l, l_r)$ will be called the critical exponent of the map. If $l_l = l_r$ the map will be referred to as symmetric.
1.2. Rotation number and combinatorics. Let \( f : S^1 \to S^1 \) be a map in \( \mathcal{L} \) and \( F : \mathbb{R} \to \mathbb{R} \) a lift of \( f \). Then

\[
\rho(F) := \lim_{n \to \infty} \frac{1}{n} (F^n(x) - x)
\]

exists for all \( x \in \mathbb{R} \).

\( \rho(F) \) is independent of \( x \) and well defined up to an integer; that is, if \( \tilde{F} \) is another lift of \( f \), then \( \rho(F) - \rho(\tilde{F}) = F - \tilde{F} \in \mathbb{Z} \). This remark justifies the following terminology:

**Definition 1.2.** Let \( F \) be a lift of \( f \). The rotation number \( \rho(f) \) of \( f \) is defined as \( \rho(F) \) modulo 1.

In the discussion that follows in this paper we will assume that the rotation number is *irrational*. Also, it will often be convenient to identify \( f \) and \( F \) and subsets of \( S^1 \) with the corresponding subsets of \( \mathbb{R} \).

The irrational rotation number \( \rho(f) \) can be written as an infinite continued fraction

\[
\rho(f) = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},
\]

where \( a_i \) are positive integers.

If we cut off the portion of the continued fraction beyond the \( n \)th position, and write the resulting fraction in lowest terms as \( \frac{p_n}{q_n} \), then the numbers \( q_n \) for \( n \geq 1 \) satisfy the recurrence relation

\[
q_{n+1} = a_{n+1} q_n + q_{n-1}; \quad q_0 = 1; \quad q_1 = a_1.
\]

The number \( q_n \) is the iterate of the rotation by \( \rho(f) \) for which the orbit of any point makes the closest return so far to the point itself.

1.3. Discussion and statement of the results. The study of nontrivial recurrent trajectories on the 2-torus has a long history, see [1,13]. It is easy to construct a continuous flow on the torus which has a non-trivial minimal set. In [3], Denjoy showed that these continuous examples can not be made analytic. In 1938, Cherry constructed analytic examples on the 2-torus with one hyperbolic saddle and sink with a non-trivial quasi-minimal set, [2]. These examples are now called Cherry flows. One of the objectives of the paper is to prove that there exist Cherry flows with metrically non-trivial quasi-minimal sets.

The proof of the main theorem of the paper goes through 1-dimensional dynamics thanks to the existence of a global Poincaré section. The first return map to the global section falls into the class \( \mathcal{L} \) of order preserving circle maps with a flat spot. Following the terminology for these flows we will distinguish three cases and we will classify the maps in \( \mathcal{L} \) depending on their critical exponent \((l, l)\):

- the non-dissipative case if \( l < 1 \);
- the conservative case if \( l = 1 \);
- the dissipative case if \( l > 1 \).

Our goal is to describe the geometric structure of the set \( \Omega \) of non-wandering points.