Addendum to: A Renormalizable 4-Dimensional Tensor Field Theory

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Abstract: This note fills a gap in the article (Ben Geloun and Rivasseau, Commun Math Phys 318:69–109, 2013). We provide the proof of Eq. (82) of Lemma 5 in Ben Geloun and Rivasseau (Commun Math Phys 318:69–109, 2013) and thereby complete its power-counting analysis with a more precise next-to-leading-order estimate.

1. Introduction

Recently, a just renormalizable tensor quantum field model in four dimensions was introduced and analyzed by the present authors [1]. This model has possible relevance for a quantum theory of gravity [2], since it effectuates in a new way a statistical sum over simplicial pseudo-manifolds in four dimensions. It has been subsequently proved asymptotically free in the ultraviolet regime [3]. From the physical point of view, this hints at a likely phase transition in the infrared regime.

The renormalization of the model followed from a multi-scale analysis and a generalized locality principle, leading to a power-counting theorem. The divergent graphs were identified, leading to the list of all marginal and relevant interactions. But it escaped our attention that one inequality (Eq. (82), see Lemma 5 in [1]) which had been used to establish this list (see Eq. (85) Lemma 6 and the discussion after Eq. (90)) had no proper proof provided.\textsuperscript{1}

In this addendum, we close this gap by providing a full-fledged proof of Eq. (82) and, in fact, an improved bound of the same form. Therefore [1] and all subsequent papers hold without change.

For this purpose, our new inequality, written in the notations of [1] Sect. 5, is given by:
Proposition 1. The degree of a rank-4 uncolored tensor graph with $\sum J_0 g_{J_0} = q > 0$ satisfies

$$\sum_J g_J - 4q \geq 6p, \quad (1)$$

where $p \in \mathbb{N}$ and $p \geq q/3$. In particular $p \geq 1$, as claimed in Eq. (82) of [1].

In the rest of this note, we establish some general lemmas valid for tensor graphs of any rank and which hold jacket by jacket. Proposition 1 is then deduced as a special case of a whole set of similar results that could be established with these lemmas. In fact, this note is therefore also a first step in a possible future systematic study of $1/N$-subdominant contributions in the wider context of general uncolored [5] tensor models [6] and tensor group field theories [1–4, 7, 8] (see also [9] for related results in the colored context).

2. Deletion and Contraction Moves on Graphs

In this section, we recall some facts about uncolored tensor graphs $G$ of rank $D$, which have $D$-stranded lines of color 0, and external half-lines (also of color 0). They can be related by a one to one correspondence to $D + 1$ colored tensor graphs $G_{\text{color}}$ [1, 5, 10] (see, in particular, Def. 1 in [1]), which have $D$-stranded lines of colors 0, 1, … $D$. The lines of color 1, … $D$ will be called below the colored lines, and the 0-lines will be also called white lines. We shall establish some properties of these graphs under contraction or deletion of these white lines.

Let us recall that the dominant graphs of the tensor $1/N$ expansion are called melonic graphs or simply melons [6, 11]. In a melon, every vertex $x$ has a single mirror vertex $\tilde{x}$ such that these two vertices are connected by $D$ two-point functions, each of a different color (see Fig. 1).

The first lemma below is already known in the context of ribbon graphs and of colored tensor graphs [6] but not enunciated in the uncolored context (which is the relevant one here).

We start by recalling the definition of tree contraction.

Definition 1. Given a connected (uncolored) tensor graph $G$ and a tree $T$ of white lines, contracting the tree leads to a reduced graph $G_1 = G/T$ called a (tensor) “rosette”.\(^2\) It has a single vertex which is a $D$-colored tensor graph plus white loop lines and external lines hooked to that vertex.

Lemma 1. We have, for any connected $G$ and $T$,

$$g_J = g_{J_1}, \quad g_{J_0} = g_{J_01}. \quad (2)$$

Remark 1. We use the obvious notations where the subscript 1 means that the quantities are computed in the tensor rosette graph $G_1 = G/T$. The result therefore holds for any pinched jacket $\tilde{J}$ of $G_{\text{color}}$ associated with $G$ yielding after contraction the jacket $\tilde{J}_1$ of $G_1 = G/T$, and any boundary jacket $J_0$ of the boundary graph $\partial G$ yielding after contraction the boundary jacket $J_01$ of the boundary of $G_1$.

\(^2\) In [12], contracting successively a tensor rosette, with respect to a tree of lines in each color, yields the so-called “core graph”. In a sense, the contraction of a tree of white line can be called a “pre” or “0-core” graph. Inspired by the ribbon graph case, we use the name of rosette also in the tensor case.