Renormalization Horseshoe and Rigidity for Circle Diffeomorphisms with Breaks

Konstantin Khanin\(^1\), Alexey Teplinsky\(^2\)

\(^1\) University of Toronto, 40 St. George Street, Toronto, Ontario M5S 2E4, Canada. E-mail: khanin@math.toronto.edu
\(^2\) Institute of Mathematics NASU, 3 Tereshchenkivska Street, Kiev 01601, Ukraine. E-mail: teplinsky@imath.kiev.ua

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Abstract: In this paper we build the renormalization horseshoe for the circle homeomorphisms, which are \(C^{2+\alpha}\)-smooth everywhere except for one point, and at that point have a jump in first derivative. We also show that two such homeomorphisms are \(C^1\)-smoothly conjugate for a certain class of rotation numbers, which include non-Diophantine numbers with arbitrarily high rate of growth.

1. Introduction

The main aim of this paper is to present an almost complete renormalization theory for circle homeomorphisms with break points. Nonlinear circle homeomorphisms with breaks were introduced about 20 years ago. Those are homeomorphisms which are smooth everywhere except at a single point where the first derivative has a jump discontinuity. The motivation was mainly based on a rich renormalization behavior such maps exhibit, having many properties normally associated with “criticality”: singularity of the invariant measure, nontrivial scalings, prevalence of periodic trajectories in one-parameter families etc (see [1] for more details). Another motivation is related to a recent interest in so-called generalized, or nonlinear, interval exchange transformations [2]. It is well-known that circle rotation can be considered as interval exchange of two intervals. The “generalized” version of it will be a circle homeomorphism. While matching conditions for the images of the end points are natural, the matching conditions on the derivatives are not. Thus one ends up with a circle homeomorphism with two break points. However, since two break points belong to the same trajectory, the map can be easily conjugated to a break map with exactly one break point. This shows that break homeomorphisms can be considered as first non-trivial examples of generalized interval exchanges. While rigidity analysis in the break case can provide certain intuition for the case of “genuine” interval exchanges, we must also emphasize a significant difference. Namely, two piecewise-smooth irrational circle homeomorphisms are topologically conjugate provided that they are combinatorially equivalent. This is in general not true for
interval exchange transformations [3]. In other words, Denjoy’s theory holds only in the case of circle homeomorphisms.

Renormalization approach, which plays a central role in the modern theory of dynamical systems, can be considered as the main tool in establishing rigidity results. The rigidity theory aims at proving the smoothness of conjugacy between two dynamical systems, which a priori are only topologically equivalent. In the context of circle dynamics it means that two maps with the same irrational rotation number and the same local structure of their singular points must be smoothly conjugate to each other. In most cases one also has to impose certain Diophantine conditions on the rotation number. The main step in proving rigidity is convergence of renormalizations. Herman’s theory [4] deals with the classical case of smooth diffeomorphisms, where renormalizations approach a family of linear maps with slope 1. On the other end, a highly nontrivial renormalization behavior was discovered for critical circle maps. There is strong numerical evidence that renormalizations behave universally for maps with the same order of critical points and the same irrational rotation number, although at present rigorous results are available only in the case when the order of the critical points is an odd integer ([5,6]). By a rather standard argument, the convergence of renormalizations implies the smoothness of conjugacy for rotation numbers of bounded type, and can be extended further to a broader class of rotation numbers. In the case of $C^\infty$-smooth circle diffeomorphisms this class consists of Diophantine numbers [7]. It is interesting that in the case of critical circle maps the $C^1$-rigidity holds for all irrational rotation numbers [8].

Circle maps with breaks form another interesting setting. In this case the local structure of the break point is determined by fixing the ratio of left and right derivatives at it. This parameter, which is obviously invariant under smooth changes of coordinates, play the same role as the order of critical point for critical circle maps. Namely, one can expect that renormalizations of two maps with the same irrational rotation number and the same ratio of the corresponding left and right derivatives, are getting exponentially close to each other. It was known for a long time [9] that in the case of maps with breaks renormalizations converge to a two-parameter family of linear-fractional maps. This essentially reduces analysis to a study of renormalizations for this canonical family. Such an analysis is the main result of the present paper. We prove that the corresponding two-dimensional transformation has strong hyperbolic properties, which allows us to construct the full renormalization horseshoe. Previously, hyperbolicity has been proved only in the case of rotation numbers with periodic continued fraction expansion [1]. Such maps correspond to periodic orbits for renormalizations. Here we prove that maps with a given irrational rotation number form a stable manifold in terms of renormalizations. Our approach is based on two ingredients. We first construct cones containing the stable manifolds, and then use the exact symmetry to prove uniform convergence of renormalizations. Finally, we prove that under certain technical conditions on the rotation numbers renormalizations for general diffeomorphisms with breaks converge to the renormalization horseshoe, which implies rigidity for the corresponding rotation numbers.

The structure of this paper is following. In Sect. 2 we present the basic renormalization construction for orientation-preserving circle homeomorphisms and, more generally, for so-called commuting pairs. In Sect. 3 we define circle diffeomorphisms with breaks and describe the properties of their renormalizations. Section 4 gives the complete description of the uniformly hyperbolic horseshoe in the two-dimensional invariant manifold consisting of linear-fractional commuting pairs, to which the sequences of renormalizations converge. In Sect. 5 we fit together the hyperbolic dynamics inside the linear-fractional manifold and the convergence of renormalizations to that manifold in order to prove the