Abstract: Given an irreducible unitary representation of a cocompact lattice of \( \text{SL}(2, \mathbb{C}) \), we explicitly write down a solution of the Strominger system of equations. These solutions satisfy the equation of motion, and the underlying holomorphic vector bundles are stable.

1. Introduction

Evoking physical requirements from anomaly cancellations, realistic fermionic spectrum and the appropriate amount \((N = 1)\) of Space-time supersymmetry, Candelas et al. had originally proposed a model for compactification of the superstring, by analyzing the vacuum configurations of these 10-dimensional theories [CHSW]. Anomaly cancellation requirements (which constrain the gauge groups of these models to be \( O(32) \) or \( E_{8} \times E_{8} \), along with the requirement of a zero cosmological constant, then lead them to propose/construct the 10-dimensional vacuum solutions of these theories to be of the metric product type \( X_{4} \times M \), where \( X_{4} \) is the maximally symmetric 4d space-time (which should admit unbroken \( N = 1 \) supersymmetry), and \( M \) is a complex 3-dimensional Calabi-Yau manifold. Subsequently, these conclusions were further generalized to include other gauge groups (like \( \text{SU}(4) \) or \( \text{SU}(5) \)), as would arise when considering compactifications for the strongly coupled heterotic string theory. The correspondence between the algebro-geometric notion of stable vector bundles and the existence of Hermitian-Yang-Mills connections was one of the primary mathematical inputs underlying these derivations [Wi]. In all these examples, the supersymmetric vacuum (manifold) was assumed to be one whose geometry had no torsion. Hence the existence of a solution on such a given manifold was mostly a topological question and the issue of existence of appropriate solutions (obeying all the physical requirements) often boiled down to a set of conditions on the Chern classes of the vacuum manifold \( M \) and the Yang-Mills Gauge connections.
In 1986, Strominger investigated the necessary and sufficient conditions for space-time supersymmetric solutions of the heterotic string. While considering more general space-times as solutions to the heterotic superstring solutions, Strominger, [St], was lead to considering vacuum configurations with torsion. He relaxed the requirement of the 10-dimensional vacuum metric by considering that, for more general vacuum configurations (which can sustain non-zero fluxes as well as space-time supersymmetry), the 10-dimensional space-time be a warped product of $X_4$ and the 6-dimensional internal space $M$. Analyzing the constraints imposed by the requirements of $N = 1$ space-time (i.e., 4 dimensional) supersymmetry (and other usual consistency requirements like anomaly cancellation), Strominger then established that the 6-dimensional internal manifold $M$ should be a compact, connected, complex manifold (hereafter denoted as $M$), such that its canonical line bundle $K_M$ is holomorphically trivial. Let $\omega = \sqrt{-1} g_{ij} dz^i \wedge dz^j$ be a $(1, 1)$ Hermitian form on $M$, and let $\nabla^M$ be a connection on $TM$ compatible with $\omega$. We denote its curvature by $R$. Further, let $E$ be a holomorphic vector bundle on $M$ equipped with the (gauge) connection $A$, and corresponding curvature $F_A$. It turns out that the anomaly cancellation condition then demands that the Hermitian $(1, 1)$ form $\omega$ obeys an equation of the form:

$$\sqrt{-1} \partial \bar{\partial} \omega = \frac{g'}{4} \left( \text{trace}(R \wedge R) - \text{trace}(F_A \wedge F_A) \right).$$

The consistency conditions from requirements of the space-time supersymmetry translates into the equation:

$$d^* \omega = \sqrt{-1} \left( \bar{\partial} - \partial \right) \ln \| \Omega \|_\omega$$

for the Hermitian form $\omega$ and the holomorphic 3-form $\Omega$. The previous equation may also be equivalently re-written as [LY2]:

$$d \left( \| \Omega \|_\omega \cdot \omega^2 \right) = 0.$$

The above equations, along with the system (constraining the Yang-Mills Gauge theory content):

$$F^2_{A,0} = F^0_{A,2} = 0, \quad F \wedge \omega^2 = 0$$

gives a complete and general solution of a superstring theory with torsion and with a flux that allows a non-trivial dilation field (cosmological constant). Henceforth, the above system of equations (which are derived solely from the explicit requirements stemming from Superstring theory) would be referred to as the Strominger system of equations. Thus, by considering vacuum geometries with torsion, Strominger was able to relax the requirement of $M$ to be Kähler and consider more general complex 3-manifolds. But the price to be paid was that the familiar tools and methods from Kähler geometry could now no longer be applied to these more general cases. Moreover, a purely topological characterization and classification of these heterotic superstring vacua solutions (i.e., the Chern classes of the bundles $E$ and the vacuum manifold $M$), would no longer suffice.

The above results provide us with the necessary and sufficient conditions for any heterotic superstring theory solution (admitting space-time supersymmetry for its vacuum configuration) to exist, but in practice, it is quite a difficult matter to exhibit or actually explicitly construct a solution which exists (and satisfies the Strominger equation). Apart from its interest and usefulness in the context of string theory, it is also of interest from