Rationality of Bershadsky-Polyakov Vertex Algebras

Tomoyuki Arakawa

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.
E-mail: arakawa@kurims.kyoto-u.ac.jp

Received: 22 July 2012 / Accepted: 30 December 2012
Published online: 9 August 2013 – © Springer-Verlag Berlin Heidelberg 2013

Abstract: We prove the conjecture of Kac-Wakimoto on the rationality of exceptional \( W \)-algebras for the first non-trivial series, namely, for the Bershadsky-Polyakov vertex algebras \( W_{3}^{(2)} \) at level \( k = p/2 - 3 \) with \( p = 3, 5, 7, 9, \ldots \). This gives new examples of rational conformal field theories.

1. Introduction

Recently, a remarkable family of \( W \)-algebras associated with simple Lie algebras and their non-principal nilpotent elements, called exceptional \( W \)-algebras, has been discovered by Kac and Wakimoto [10]. In [10] it was conjectured that with an exceptional \( W \)-algebra one can associate a rational conformal field theory.

As a first step to resolve the Kac-Wakimoto conjecture we have proved in the previous article [3] that exceptional \( W \)-algebras are lisse, or equivalently [2], \( C_2 \)-cofinite. Therefore it remains [6,15] to show that exceptional \( W \)-algebras are rational, i.e., that the representations are completely reducible, in order to prove the conjecture. In this article we prove the rationality of the first non-trivial series of exceptional \( W \)-algebras, that is, the Bershadsky-Polyakov (vertex) algebras \( W_{3}^{(2)} \) [4,13] at level \( k = p/2 - 3 \) with \( p = 3, 5, 7, 9, \ldots \). The vertex algebra \( W_{3}^{(2)} \) is the \( W \)-algebra associated with \( g = \mathfrak{sl}_3 \) and its minimal nilpotent element.

Let us state our main result more precisely: Let \( \mathcal{W}_k \) denote the unique simple quotient of \( W_{3}^{(2)} \) at level \( k \neq -3 \).

Main Theorem (Conjectured by Kac and Wakimoto [10]). Let \( p \) be an odd integer equal or greater than 3, \( k = p/2 - 3 \). Then the vertex algebra \( \mathcal{W}_k \) is rational. The simple \( \mathcal{W}_k \)-modules are parameterized by the set of integral dominant weights of \( \mathfrak{sl}_3 \) of

\* This work is partially supported by the JSPS Grant-in-Aid for Scientific Research (B) No. 20340007.
level $p - 3$. These simple modules can be obtained by the quantum BRST reduction from irreducible admissible representations of $\hat{sl}_3$ of level $k$.

For $p = 3$, $\mathcal{W}_{3/2-3}$ is one-dimensional. In the remaining cases $\mathcal{W}_{p/2-3}$ are conformal with negative central charges.

We note that Zhu’s algebra of $\mathcal{W}_3^{(2)}$ is closely related with Smith’s algebra [14] which is a deformation of the universal enveloping algebra $U(sl_2(\mathbb{C}))$ of $sl_2(\mathbb{C})$, and that the rational quotient $\mathcal{W}_{p/2-3}$ has features in common with the $sl_2$-integrable affine vertex algebras in the sense that the following relations hold:

$$G^+(z) = G^-(z) = 0,$$

where $G^+(z)$ and $G^-(z)$ are the standard generating fields of $\mathcal{W}_{p/2-3}$, see below.

2. Bershadsky-Polyakov Algebras at Exceptional Levels

Let $\mathcal{W}^k$ denote the Bershadsky-Polyakov (vertex) algebra $\mathcal{W}_3^{(2)}$ at level $k \neq -3$, which is the vertex algebra freely generated by the fields $J(z), G^\pm(z), T(z)$ with the following OPE’s:

$$J(z)J(w) \sim \frac{2k + 3}{3(z - w)^2}, \quad G^\pm(z)G^\pm(w) \sim 0,$$

$$J(z)G^\pm(w) \sim \pm \frac{1}{z - w} G^\pm(w),$$

$$T(z)T(w) \sim -\frac{(2k + 3)(3k + 1)}{2(k + 3)(z - w)^4} + \frac{2}{(z - w)^2} T(w) + \frac{1}{z - w} \partial T(w),$$

$$T(z)G^\pm(w) \sim \frac{3}{2(z - w)^2} G^\pm(w) + \frac{1}{z - w} \partial G^\pm(w),$$

$$T(z)J(w) \sim \frac{1}{(z - w)^2} J(w) + \frac{1}{z - w} \partial J(w),$$

$$G^+(z)G^-(w) \sim \frac{(k + 1)(2k + 3)}{(z - w)^3} + \frac{3(k + 1)}{(z - w)^2} J(w) + \frac{1}{z - w} \left(3 : J(w)^2 : + \frac{3(k + 1)}{2} \partial J(w) - (k + 3)T(w)\right).$$

As in the Introduction we denote by $\mathcal{W}_k$ the unique simple quotient of $\mathcal{W}^k$.

**Theorem 2.1 ([3]).** Let $k, p$ be as in the Main Theorem. Then $\mathcal{W}_k$ is lisse, or equivalently, $C_2$-cofinite.

Set

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = T(z) + \frac{1}{2} \partial J(w).$$

This defines a conformal vector of $\mathcal{W}^k$ with central charge

$$c(k) = -\frac{4(k + 1)(2k + 3)}{k + 3} = -\frac{4(p - 4)(p - 3)}{p}.$$