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Abstract: We prove a unique continuation principle for spectral projections of Schrödinger operators. We consider a Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$, and let $H_\Lambda$ denote its restriction to a finite box $\Lambda$ with either Dirichlet or periodic boundary condition. We prove unique continuation estimates of the type $\chi_I(H_\Lambda) W \chi_I(H_\Lambda) \geq \kappa \chi_I(H_\Lambda)$ with $\kappa > 0$ for appropriate potentials $W \geq 0$ and intervals $I$. As an application, we obtain optimal Wegner estimates at all energies for a class of non-ergodic random Schrödinger operators with alloy-type random potentials (‘crooked’ Anderson Hamiltonians). We also prove optimal Wegner estimates at the bottom of the spectrum with the expected dependence on the disorder (the Wegner estimate improves as the disorder increases), a new result even for the usual (ergodic) Anderson Hamiltonian. These estimates are applied to prove localization at high disorder for Anderson Hamiltonians in a fixed interval at the bottom of the spectrum.

1. introduction

Let $H = -\Delta + V$ be a Schrödinger operator on $L^2(\mathbb{R}^d)$. Given a box (or cube) $\Lambda = \Lambda_L(x_0) \subset \mathbb{R}^d$ with side of length $L$ and center $x_0 \in \mathbb{R}^d$, let $H_\Lambda = -\Delta_\Lambda + V_\Lambda$ denote the restriction of $H$ to the box $\Lambda$ with either Dirichlet or periodic boundary condition: $\Delta_\Lambda$ is the Laplacian with either Dirichlet or periodic boundary condition and $V_\Lambda$ is the restriction of $V$ to $\Lambda$. (We will abuse the notation and simply write $V$ for $V_\Lambda$, i.e., $H_\Lambda = -\Delta_\Lambda + V$ on $L^2(\Lambda)$.) By a unique continuation principle for spectral projections (UCPSP) we will mean an estimate of the form

$$\chi_I(H_\Lambda) W \chi_I(H_\Lambda) \geq \kappa \chi_I(H_\Lambda),$$

where $\chi_I$ is the characteristic function of an interval $I \subset \mathbb{R}$, $W \geq 0$ is a potential, and $\kappa > 0$ is a constant.

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If $V$ and $W$ are bounded $\mathbb{Z}^d$-periodic potentials, $W \geq 0$ with $W > 0$ on some open set, Combes, Hislop and Klopp [CHK1, Sect. 4], [CHK2, Thm. 2.1] proved a UCSPSP for $H_\Lambda$ with periodic boundary condition, for boxes $\Lambda = \Lambda_L(x_0) \subset \mathbb{R}^d$ with $L \in \mathbb{N}$ and $x_0 \in \mathbb{Z}^d$ and arbitrary bounded intervals $I$, with a constant $\kappa > 0$ depending on $d, I, V, W$ but not on the box $\Lambda$. Their proof uses the unique continuation principle and Floquet theory. Germinet and Klein [GK4, Thm. A.6] proved a modified version of this result, using Bourgain and Kenig’s quantitative unique continuation principle [BK, Lem. 3.10] and Floquet theory, obtaining control of the constant $\kappa$ in terms of the relevant parameters.

Rojas-Molina and Veselić recently proved “scale-free unique continuation estimates” for Schrödinger operators [RV, Thm. 2.1] (see also [R2, Thm. A.1.1]). They consider a Schrödinger operator $H = -\Delta + V$, where $V$ is only required to be bounded, and its restrictions $H_\Lambda$ to boxes $\Lambda$ with side $L \in \mathbb{N}$ with either Dirichlet or periodic boundary condition. They decompose the box $\Lambda$ into unit boxes, and for each unit box pick a ball of (a fixed) radius $\delta$ contained in the unit box, and let $W$ be the potential given by the sum of the characteristic functions of those balls. Using a version of the quantitative unique continuation principle [RV, Thm. 3.1], they prove that if $\psi$ is an eigenfunction of $H_\Lambda$ with eigenvalue $E$ (more generally, if $|\Delta \psi| \leq |(V - E) \psi|$), then

$$\|W \psi\|_2^2 \geq \kappa \|\psi\|_2^2,$$

(1.2)

where the constant $\kappa > 0$ depends only on $d, V, \delta, E$, and is locally bounded on $E$. Since (1.2) is just the UCSPSP (1.1) when $I = \{E\}$, this raises the question of the validity of a UCSPSP in this setting, posed as an open question by Rojas-Molina and Veselić [RV].

In this article we prove a UCSPSP for Schrödinger operators (Theorem 1.1), giving an affirmative answer to the open question in [RV]. The proof is based on the quantitative unique continuation principle derived by Bourgain and Klein [BK1, Thm. 3.2], restated here as Theorem 2.1. This version of the quantitative unique continuation principle, as the original result of Bourgain and Kenig [BK, Lem. 3.10] and the version of Germinet and Klein [GK4, Thm. A.1], allows for approximate solutions of the stationary Schrödinger equation. ([RV, Thm. 3.1] requires $|\Delta \psi| \leq |V \psi|$.) Theorem 2.1 can be applied not only to eigenfunctions of a Schrödinger operator $H$, but also to approximate eigenfunctions, i.e., arbitrary $\psi \in \text{Ran} \chi_{[E-\gamma, E+\gamma]}(H)$, with the error controlled by $\|(H - E) \psi\|_2 \leq \gamma \|\psi\|_2$. (See the derivation of [GK4, Thm. A.6] from [GK4, Thm. A.1].) The notion of “dominant boxes”, introduced by Rojas-Molina and Veselić [RV, Subsect. 5.2] (see also [R2, App. A]), plays an important role in the derivation of Theorem 1.1 from Theorem 2.1.

Using Theorem 1.1, we obtain (Theorems 1.4 and 1.5) optimal Wegner estimates (i.e., with the correct dependence on the volume and interval length) at all energies for a class of non-ergodic random Schrödinger operators with alloy-type random potentials (called crooked Anderson Hamiltonians in Definition 1.2). As a consequence, we get optimal Wegner estimates for Delone-Anderson models at all energies (Remark 1.6). We also prove (Theorem 1.7) optimal Wegner estimates at the bottom of the spectrum for crooked Anderson Hamiltonians that have the expected dependence on the disorder (in particular, the Wegner estimate improves as the disorder increases), a new result even for the usual (ergodic) Anderson Hamiltonian. Using Theorem 1.7, we prove localization at high disorder for Anderson Hamiltonians in a fixed interval at the bottom of the spectrum (Theorem 1.8); such a result was previously known only with a covering condition [GK2, Thm. 3.1].