Conification of Kähler and Hyper-Kähler Manifolds

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Abstract: Given a Kähler manifold $M$ endowed with a Hamiltonian Killing vector field $Z$, we construct a conical Kähler manifold $\hat{M}$ such that $M$ is recovered as a Kähler quotient of $\hat{M}$. Similarly, given a hyper-Kähler manifold $(M, g, J_1, J_2, J_3)$ endowed with a Killing vector field $Z$, Hamiltonian with respect to the Kähler form of $J_1$ and satisfying $\mathcal{L}_Z J_2 = -2J_3$, we construct a hyper-Kähler cone $\hat{M}$ such that $M$ is a certain hyper-Kähler quotient of $\hat{M}$. In this way, we recover a theorem by Haydys. Our work is motivated by the problem of relating the supergravity c-map to the rigid c-map. We show that any hyper-Kähler manifold in the image of the c-map admits a Killing vector field with the above properties. Therefore, it gives rise to a hyper-Kähler cone, which in turn defines a quaternionic Kähler manifold. Our results for the signature of the metric and the sign of the scalar curvature are consistent with what we know about the supergravity c-map.

Introduction

Let us recall that there is an interesting geometric construction called the c-map, which was found by theoretical physicists. There are in fact two versions of the c-map: the supergravity c-map and the rigid c-map. The supergravity c-map associates a quaternionic Kähler manifold of negative scalar curvature with any projective special Kähler manifold, see [FS,H2,CM]. The metric is explicit but rather complicated. The rigid c-map is much simpler, see [CFG,H1,ACD]. It associates a hyper-Kähler manifold with any affine special Kähler manifold. The initial motivation for this work was our idea to reduce the supergravity c-map to the rigid c-map by means of a conification of the hyper-Kähler manifold obtained from the rigid c-map. Let us explain this idea in more detail.

Since any projective special Kähler manifold $\hat{M}$ is the base of a $\mathbb{C}^\ast$-bundle with the total space a conical affine special Kähler manifold $M$, we have the following diagram:
where $c$ stands for the rigid c-map, $\tilde{c}$ for the supergravity c-map and $N$, $\tilde{N}$ are the resulting (pseudo-)hyper-Kähler and quaternionic Kähler manifolds, respectively. We have indicated the real dimension. This shows that $N$ cannot simply be the Swann bundle $\hat{N}$ over $\tilde{N}$. In fact, $N$ is in general not conical and the (pseudo-)hyper-Kähler cone $\hat{N}$ should be obtained from $N$ by a certain conification procedure $N^{4n} \xrightarrow{\text{con}} \hat{N}^{4n+4}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
M^{2n} & \xrightarrow{c} & N^{4n} \\
\downarrow \quad C^* & & \downarrow \quad \mathbb{H}^*/\pm 1 \\
\tilde{M}^{2n-2} & \xrightarrow{\tilde{c}} & \tilde{N}^{4n}
\end{array}
$$

We are also interested in the analogous problem for the $r$-map, where we have a diagram of the form:

$$
\begin{array}{ccc}
M^n & \xrightarrow{r} & N^{2n} \\
\downarrow \quad \mathbb{R}^{>0} & & \downarrow \quad C^* \\
\tilde{M}^{n-1} & \xrightarrow{\tilde{r}} & \tilde{N}^{2n}
\end{array}
$$

Now $M$ is an affine special real manifold with homogeneous cubic prepotential, $\tilde{M}$ is the corresponding projective special real manifold, $r$ is the rigid $r$-map [CMMS1, AC], $\tilde{r}$ is the supergravity $r$-map [DV, CM] and $\hat{N}$ is the conical affine special Kähler manifold over the projective special Kähler manifold $\tilde{N}$.

An important inspiration for our work has been the paper [Haydys] by Haydys, see also [APP] in which Haydys construction is called the QK/HK correspondence. The construction has two parts. The first part is the hyper-Kähler reduction of a hyper-Kähler cone with respect to a Hamiltonian Killing vector field which is compatible with the cone structure. The hyper-Kähler manifold $(M, g, J_1, J_2, J_3)$ obtained by such a reduction inherits a Killing vector field $Z$ which preserves one of the three complex structures $J_1$ of the hyper-Kähler triplet $(J_\alpha)$ and rotates the two other ones. The second part is the inversion of the reduction, which is much more involved than the first part. As a result of our careful analysis, we are able to give our own proof of the inversion recovering and extending the results by Haydys. Under the assumptions stated precisely in Sect. 2, the conical hyper-Kähler structure is rigorously established in Theorem 2. The final formulas are explicit enough to allow for further progress in the study of hyper-Kähler manifolds obtained by such a conification. As an example, we can easily compute the signature and scalar curvature of the resulting quaternionic Kähler manifolds, see Corollary 1 and Corollary 2. These results are new even in the case when the initial hyper-Kähler metric is positive definite, as considered in [Haydys]. We show that (positive definite) quaternionic Kähler manifolds of negative scalar curvature can be obtained from indefinite as well as from positive definite hyper-Kähler manifolds, whereas quaternionic Kähler manifolds of positive scalar curvature do always require a positive definite initial metric.