Construction of Hadamard States by Pseudo-Differential Calculus

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Abstract: We give a new construction based on pseudo-differential calculus of quasi-free Hadamard states for Klein–Gordon equations on a class of space-times whose metric is well-behaved at spatial infinity. In particular on this class of space-times, we construct all pure Hadamard states whose two-point function (expressed in terms of Cauchy data on a Cauchy surface) is a matrix of pseudo-differential operators. We also study their covariance under symplectic transformations.

As an aside, we give a new construction of Hadamard states on arbitrary globally hyperbolic space-times which is an alternative to the classical construction by Fulling, Narcowich and Wald.

1. Introduction

1.1. Hadamard states. Hadamard states are nowadays widely accepted as possible physical states of the non-interacting quantum field theory on a curved space-time. One of the main reasons is their applicability to renormalization of the stress-energy tensor, a necessary step in the formulation of semi-classical Einstein equations. Moreover, the Hadamard condition plays an essential role in the perturbative construction of interacting quantum field theory [BF]. Other related concepts making use of Hadamard states include local thermal equilibrium [SV] and quantum energy inequalities [FV].

Since the work of Radzikowski [R], the Hadamard condition (renamed microlocal spectrum condition), is formulated as a requirement for the wave front set of the associated two-point function $\Lambda$, which is necessarily a bi-solution of the free equations of motion. It is therefore natural to try to construct such states using the standard apparatus of microlocal analysis, based on pseudo-differential calculus. Although a construction is already known for space-times with compact Cauchy surface [J1], it does certainly not cover many cases of physical interest and lacks the capability to produce many states on a fixed space-time with distinct properties.

In this paper we address these questions for a class of space-times whose metric components are suitably well-behaved at spatial infinity, allowing also for external
potentials. In this case it is possible to obtain rather complete and transparent results. Namely we construct a large class of quasi-free Hadamard states, whose two-point functions, expressed in terms of Cauchy data on a fixed Cauchy surface, are matrices of pseudo-differential operators. In particular we can construct all such pure Hadamard states and study their covariance under symplectic transformations.

As an additional result we give a new construction of Hadamard states on arbitrary globally hyperbolic space-times which is an alternative to the classical construction by Fulling et al. [FNW]. Our method turns out not to produce pure states in general, it allows however to keep track of their local properties.

1.2. Methods. Our analysis is set on three levels:

(1) Our starting point are normally hyperbolic operators on $\mathbb{R} \times \mathbb{R}^d$ of the form

$$\partial^2_t + a(t, x, D_x) = \partial^2_t - \partial_{xj} a^{jk} \partial_{xk} + b^j \partial_{xj} - \partial_{xj} b^j + m, \quad (1.1)$$

where

$$a^{jk}, b^j, m \in C^\infty(\mathbb{R}, C^\infty_{bd}(\mathbb{R}^d)), \quad m(x) \in \mathbb{R}, \quad [a^{jk}] (x) \geq c(t) \mathbb{1} \text{ uniformly on } \mathbb{R}^{1+d}, \quad c(t) > 0. \quad (1.2)$$

We refer to this case as the model Klein–Gordon equation and give a construction of the associated parametrix for the Cauchy problem, in such way that the propagation of positive-frequency and negative-frequency singularities is under control. This allows us to reformulate the microlocal spectrum condition in terms of Cauchy data. We show how to construct many non-necessarily pure Hadamard states and then characterize pure ones. We also describe classes of symplectic transformations which preserve the microlocal spectrum condition.

(2) The above results are easily extended to operators of the form $f(\partial^2_t + a(t)) g$, where $f$ and $g$ are smooth densities. This way, we show that the problem of constructing Hadamard states is reduced to the model case above if $M = \mathbb{R} \times \mathbb{R}^d$, the metric is given by

$$g = -c(x) dt^2 + h_{jk}(x) dx^j dx^k, \quad (1.3)$$

and the Klein–Gordon operator is of the form

$$P(x, D_x) = c^{-\frac{1}{2}} |h|^{-\frac{1}{2}} (\partial_t + iV) c^{-\frac{1}{2}} |h|^{\frac{1}{2}} (\partial_t + iV)$$

$$-c^{-\frac{1}{2}} |h|^{-\frac{1}{2}} (\partial_j + iA_j) c^{\frac{1}{2}} |h|^{\frac{1}{2}} h^{jk} (\partial_k + iA_k) + \rho, \quad (1.4)$$

where $A_{\mu}(x) = (V(x), A_j(x))$, $|h| = \det[h_{jk}]$, $[h^{jk}] = [h_{jk}]^{-1}$ and the following hypotheses are assumed:

$$\forall \ I \subset \mathbb{R} \text{ compact interval } \exists \ C > 0 \text{ such that }$$

$$C \leq c(x), \ C \mathbb{1} \leq [h_{jk}(x)], \text{ uniformly for } x \in I \times \mathbb{R}^d, \quad (1.4)$$

$$h_{jk}(x), \ c(x), \ \rho(x), \ A_{\mu}(x) \in C^\infty(\mathbb{R}, C^\infty_{bd}(\mathbb{R}^d)).$$

(3) For arbitrary space-times (and external potentials), using a suitable partition of unity, we explain how to glue together two-point functions of Hadamard states on smaller regions of the space-time into a globally-defined one. Using the results obtained for the special case above, this yields a new construction of Hadamard states on arbitrary globally-hyperbolic space-times.