Radiation Fields on Schwarzschild Spacetime

Dean Baskin¹, Fang Wang²

¹ Northwestern University, Evanston, USA. E-mail: dbaskin@math.northwestern.edu
² Shanghai Jiao Tong University, Shanghai, China. E-mail: fangwang1984@sjtu.edu.cn

Received: 31 May 2013 / Accepted: 28 December 2013
Published online: 26 April 2014 – © Springer-Verlag Berlin Heidelberg 2014

Abstract: In this paper we define the radiation field for the wave equation on the Schwarzschild black hole spacetime. In this context it has two components: the rescaled restriction of the time derivative of a solution to null infinity and to the event horizon. In the process, we establish some regularity properties of solutions of the wave equation on the spacetime. In particular, we prove that the regularity of the solution across the event horizon and across null infinity is determined by the regularity and decay rate of the initial data at the event horizon and at infinity. We also show that the radiation field is unitary with respect to the conserved energy and prove support theorems for each piece of the radiation field.

1. Introduction

In this paper we define the radiation field for the wave equation on the Schwarzschild black hole spacetime. The radiation field is a rescaled restriction of the time derivative of a solution and in this case has two components: one corresponding to null infinity and one corresponding to the event horizon. In the process, we establish some regularity properties of solutions of the wave equation on the spacetime. In particular, we prove that the regularity of the solution across the event horizon and across null infinity is determined by the regularity and decay rate of the initial data at the event horizon and at infinity. We further show that the radiation field is unitary with respect to the conserved energy and prove support theorems for each component of the radiation field.

The radiation field for a solution of the wave equation describes the radiation pattern seen by distant observers. On Minkowski space $\mathbb{R} \times \mathbb{R}^n$, it is the rescaled restriction of a solution to null infinity. More precisely, one introduces polar coordinates $(r, \omega)$ in the spatial variables as well as the “lapse” parameter $s = t - r$. The forward radiation field...
of a solution $u$ of $(\partial_t^2 - \Delta) u = 0$ with smooth, compactly supported initial data is given by

$$
\lim_{r \to \infty} \partial_t r^{\frac{n-1}{2}} u(s + r, r\omega).
$$

The map taking the initial data to the radiation field of the corresponding solution provides a unitary isomorphism from the space of finite energy initial data to $L^2(\mathbb{R}^s \times S^{n-1}_\omega)$. The radiation field is a translation representation of the wave group and was initially defined by Friedlander [Fri80], though it is implicit in the work of Lax–Phillips (e.g., [LP89]) and Helgason (e.g., [Hel99]). Its definition, structure, and properties have been studied in a variety of geometric contexts [SB03,SB05,SBW05,SB08,BSB12], including settings of interest in general relativity [Wan13,BVW12].

We now recall the structure of the Schwarzschild black hole spacetime. (For a more thorough discussion, including many different coordinate systems, we direct the reader to the book of Hawking and Ellis [HE73] or to the lecture notes of Dafermos and Rodnianski [DR13].) The Schwarzschild spacetime is diffeomorphic to $\mathbb{R}_t \times (2M, \infty)_r \times S^2_\omega$ with Lorentzian metric given by

$$
g_S = -\left(\frac{r - 2M}{r}\right) dt^2 + \left(\frac{r}{r - 2M}\right) dr^2 + r^2 d\omega^2.
$$

Here $d\omega^2$ is the round metric on the unit sphere $S^2$.

We consider the Cauchy problem:

$$
\Box_S u = 0, \quad (u, \partial_t u)|_{t=0} = (\phi, \psi) \quad (1.1)
$$

where $\Box_S$ is the Laplace–Beltrami (D’Alembertian) operator for $g_S$:

$$
\Box_S = -\left(\frac{r}{r - 2M}\right) \partial_t^2 + \left(\frac{r - 2M}{r}\right) \partial_r^2 + \frac{1}{r^2} \Delta_\omega + \frac{2(r - M)}{r^2} \partial_r.
$$

Solutions $u$ of Eq. (1.1) possesses a conserved energy $E(t)$:

$$
E(t) = \int_{2M}^{\infty} \int_{S^2} e(t)r^2 d\omega dr,
$$

where

$$
e(t) = \left(1 - \frac{2M}{r}\right)^{-1} (\partial_t u)^2 + \left(1 - \frac{2M}{r}\right) (\partial_r u)^2 + \frac{1}{r^2} |\nabla_\omega u|^2.
$$

Observe that $e(t)$ is positive definite but is ill-behaved at $r = 2M$.

As the Schwarzschild black hole has two spatial ends, there are two ends through which null geodesics (i.e., light rays) can “escape”: the event horizon (at the $r = 2M$ end) and null infinity (at the $r = \infty$ end). (There are also “trapped” null geodesics tangent to the photon sphere $r = 3M$.) In terms of incoming Eddington–Finkelstein coordinates,

$$
(\tau = t + r + 2M \log(r - 2M), r, \omega),
$$

the event horizon corresponds to $r = 2M$. Similarly, in terms of outgoing Eddington–Finkelstein coordinates,

$$
(\tilde{\tau} = t - r - 2M \log(r - 2M), r, \omega),
$$

we have null infinity.