Hertz Potentials and Asymptotic Properties of Massless Fields

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Abstract: In this paper we analyze Hertz potentials for free massless spin-s fields on the Minkowski spacetime, with data in weighted Sobolev spaces. We prove existence and pointwise estimates for the Hertz potentials using a weighted estimate for the wave equation. This is then applied to give weighted estimates for the solutions of the spin-s field equations, for arbitrary half-integer s. In particular, the peeling properties of the free massless spin-s fields are analyzed for initial data in weighted Sobolev spaces with arbitrary, non-integer weights.

1. Introduction

The analysis by Christodoulou and Klainerman of the decay of massless fields of spins 1 and 2 on Minkowski space [9] served as an important preliminary for their proof of the non-linear stability of Minkowski space [10]. The method used in [9] was based on energy estimates using the vector fields method, see [18]. This approach was extended to fields of arbitrary spin by Shu [32]. The approach of [10] to the problem of nonlinear stability of Minkowski space was later extended by Klainerman and Nicolò [21] to give the full peeling behavior for the Weyl tensor at null infinity.

The vector fields method makes use of the conformal symmetries of Minkowski space to derive conservation laws for higher order energies, which then via the Klainerman Sobolev inequality [20] give pointwise estimates for the solution of the wave equation. An analogous procedure is used for the higher spin fields in the papers cited above. This procedure gives pointwise decay estimates for the solution of the Cauchy problem of the wave equation and the spin-s equation, for initial data of one particular falloff at spatial infinity. The conditions on the initial data originate in the growth properties of the conformal Killing vector fields on Minkowski space, which are used in the energy estimates.
Let $H^j_\delta$ be the weighted $L^2$ Sobolev spaces on $\mathbb{R}^3$, that is to say, the space of functions $\phi$ for which
\[
\sum_j \| <r>^{-(3/2+\delta)+k} D^k \phi \|^2_{L^2(\mathbb{R}^3)} < \infty, \quad \text{where} \quad <r> = (1 + r^2)^{1/2}.
\]
We use the conventions\(^1\) of Bartnik [3]. Since we shall use the 2-spinor formalism, as defined in Sect. 2.1, we work here and throughout the paper on Minkowski space with signature $+--$. Consider the Cauchy problem for the wave equation
\[
\Box \phi = 0, \quad \phi|_{t=0} = f \in H^j_{-3/2}, \quad \partial_t \phi|_{t=0} = g \in H^{j-1}_{-5/2}.
\]
Then, for $j \geq 2$, one has the estimate [18]
\[
|\phi(x, t)| \leq C <u>^{-1/2} <v>^{-1} (\| f \|_{j-3/2} + \| g \|_{j-1,-5/2}), \quad (1.2)
\]
where $<u> = (1 + u^2)^{1/2}$, $u = t - r$ and $v = t + r$. On the other hand, if one considers the wave equation (1.1) on the flat 3+1 dimensional Minkowski spacetime as a special case of the conformally covariant form of the wave equation
\[
(\nabla^a \nabla_a + R/6) \phi = 0,
\]
the condition on the initial data which is compatible with regular conformal compactification is
\[
\partial^\ell f = O(r^{-2-\ell}), \quad \partial^\ell g = O(r^{-3-\ell}). \quad (1.3)
\]
Making use of standard energy estimates in the conformal compactification of Minkowski space, one arrives, after undoing the conformal compactification, at
\[
|\phi(x, t)| = O \left( <u>^{-1} <v>^{-1} \right), \quad (1.4)
\]
see the discussion in [16, §6.7]. In particular, there is an extra $r^{-1/2}$ falloff in the condition (1.3) on the initial data compared to (1.1) as well as an additional factor $<u>^{-1/2}$ decay in the retarded time coordinate $u$ in (1.4) compared to (1.2).

Let us now consider the case of higher spin fields. Let $2s$ be a positive integer and let $\phi_{A...F}$ be a totally symmetric spinor field of valence $2s$. The Cauchy problem for a massless spin-$s$ field is
\[
\nabla_A \phi^{A}_{A...F} = 0, \quad \phi_{A...F}|_{t=0} = \varphi_{A...F}.
\]
For $s \geq 1$, the Cauchy datum $\varphi_{A...F}$ must satisfy the constraint equation
\[
D^{AB} \varphi_{AB...F} = 0,
\]
where $D_{AB}$ is the intrinsic space spinor derivative on $\Sigma$, see Sect. 2.1. The spin-$1/2$ case does not have constraints.

\(^1\) The spaces $H^j_\delta$ are in [3] denoted by $W^j_\delta$. 