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Green-Hyperbolic Operators on Globally Hyperbolic Spacetimes

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Abstract: Green-hyperbolic operators are linear differential operators acting on sections of a vector bundle over a Lorentzian manifold which possess advanced and retarded Green's operators. The most prominent examples are wave operators and Dirac-type operators. This paper is devoted to a systematic study of this class of differential operators. For instance, we show that this class is closed under taking restrictions to suitable subregions of the manifold, under composition, under taking "square roots", and under the direct sum construction. Symmetric hyperbolic systems are studied in detail.

Introduction

Green-hyperbolic operators are certain linear differential operators acting on sections of a vector bundle over a Lorentzian manifold. They are, by definition, those operators which possess advanced and retarded Green's operators. The most prominent examples are normally hyperbolic operators (wave equations) and Dirac-type operators. The reason for introducing them in [2] lies in the fact that they can be quantized; one can canonically construct a bosonic locally covariant quantum field theory for them.

The aim of the present paper is to study Green-hyperbolic operators systematically from a geometric and an analytic perspective. The underlying Lorentzian manifold must be well behaved for the analysis of hyperbolic operators. In technical terms, it must be globally hyperbolic. In the first section we collect material about such Lorentzian manifolds. We introduce various compactness properties for closed subsets and show their interrelation. These considerations will later be applied to the supports of sections.

In the second section we study various spaces of smooth sections of our vector bundle. The crucial concept is that of a support system. This is a family of closed subsets of our manifold with certain properties making it suitable for defining a good space of sections by demanding that their supports be contained in the support system. We observe a duality principle; a distributional section has support in a support system if and only

if it extends to a continuous linear functional on test sections with support in the dual support system.

Green's operators and Green-hyperbolic differential operators are introduced in the third section. We give various examples and show that the class of Green-hyperbolic operators is closed under taking restrictions to suitable subregions of the manifold, under composition, under taking "square roots", and under the direct sum construction. This makes it a large and very flexible class of differential operators to consider. We show that Green's operators are unique and that they extend to several spaces of sections. We argue that Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem.

The fourth section is devoted to extending the Green's operators to distributional sections. We show that an important analytical result for the causal propagator (the difference of the advanced and the retarded Green's operator), also holds when one replaces smooth by distributional sections.

In the last section we study symmetric hyperbolic systems over globally hyperbolic manifolds. We provide detailed proofs of well-posedness of the Cauchy problem, finiteness of the speed of propagation and the existence of Green's operators. The crucial step in these investigations is an energy estimate for the solution to such a symmetric hyperbolic system. We conclude by observing that a symmetric hyperbolic system can be quantized in two ways; one yields a *bosonic* and the other one a *fermionic* locally covariant quantum field theory.

1. Globally Hyperbolic Lorentzian Manifolds

We summarize various facts about globally hyperbolic Lorentzian manifolds. For details the reader is referred to one of the classical textbooks [5,11,12]. Throughout this article, M will denote a time oriented Lorentzian manifold. We use the convention that the signature of M is $(- + \cdots +)$. Note that we do not specify the dimension of M nor do we assume orientability or connectedness.

1.1. Cauchy hypersurfaces. A subset $\Sigma \subset M$ is called a *Cauchy hypersurface* if every inextensible timelike curve in M meets Σ exactly once. Any Cauchy hypersurface is a topological submanifold of codimension 1. All Cauchy hypersurfaces of M are homeomorphic.

If M possesses a Cauchy hypersurface then M is called *globally hyperbolic*. This class of Lorentzian manifolds contains many important examples: Minkowski space, Friedmann models, the Schwarzschild model and deSitter spacetime are globally hyperbolic. Bernal and Sánchez proved an important structural result [6, Thm. 1.1]: Any globally hyperbolic Lorentzian manifold has a *Cauchy temporal function*. This is a smooth function $t : M \to \mathbb{R}$ with past-directed timelike gradient ∇t such that the levels $t^{-1}(s)$ are (smooth spacelike) Cauchy hypersurfaces if nonempty.

1.2. Future and past. From now on let M always be globally hyperbolic. For any $x \in M$ we denote by $J^+(x)$ the set all points that can be reached by future-directed causal curves emanating from x. For any subset $A \subset M$ we put $J^+(A) := \bigcup_{x\in A} J^+(x)$. If A is closed so is $J^+(A)$. We call a subset $A \subset M$ *strictly past compact* if it is closed and there is a compact subset $K \subset M$ such that $A \subset J^+(K)$. If A is strictly past compact so