A New Continuation Criterion for the Relativistic Vlasov–Maxwell System

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Abstract: The global existence of solutions to the relativistic Vlasov–Maxwell system given sufficiently regular finite energy initial data is a longstanding open problem. The main result of Glassey and Strauss (Arch Ration Mech Anal 92:59–90, 1986) shows that a solution \((f, E, B)\) remains \(C^1\) as long as the momentum support of \(f\) remains bounded. Alternate proofs were later given by Bouchut et al. (Arch Ration Mech Anal 170:1–15, 2003) and Klainerman and Staffilani (Commun Pure Appl Anal 1:103–125, 2002). We show that only the boundedness of the momentum support of \(f\) after projecting to any two dimensional plane is needed for \((f, E, B)\) to remain \(C^1\).

1. Introduction

We consider the initial value problem for the relativistic Vlasov–Maxwell system in three dimensions. Let the particle density \(f: \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3_p \to \mathbb{R}_+\) be a non-negative function of time \(t \in \mathbb{R}\), position \(x \in \mathbb{R}^3\) and momentum \(p \in \mathbb{R}^3\) and \(E, B: \mathbb{R} \times \mathbb{R}^3_x \to \mathbb{R}^3\) be time-dependent vector fields on the position space \(\mathbb{R}^3\).

The relativistic Vlasov–Maxwell system can be written as

\[
\begin{align*}
\partial_t f + \hat{p} \cdot \nabla_x f + (E + \hat{p} \times B) \cdot \nabla_p f &= 0, \\
\partial_t E &= \nabla_x \times B - j, \quad \partial_t B = -\nabla_x \times E, \\
\nabla_x \cdot E &= \rho, \quad \nabla_x \cdot B &= 0.
\end{align*}
\]

Here we have the charge

\[
\rho(t, x) \overset{\text{def}}{=} 4\pi \int_{\mathbb{R}^3} f(t, x, p) dp.
\]

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and the current
\[ j_i(t, x) \overset{\text{def}}{=} 4\pi \int_{\mathbb{R}^3} \hat{p}_i f(t, x, p) dp, \quad (i = 1, 2, 3). \]

In these expressions we define
\[ \hat{p} \overset{\text{def}}{=} \frac{p}{p_0}, \quad p_0 \overset{\text{def}}{=} \sqrt{1 + |p|^2}. \]

Notice that given initial data \( f_0, E_0, B_0 \) which satisfy the constraint equations (3), they are propagated by the evolution equation (1) and (2) as long as the solution remains sufficiently regular.

According to the relativistic Vlasov–Maxwell system (1)–(3), the particle density \( f \) is transported along the characteristics \((X(t), V(t))\), which verify the following ordinary differential equations:
\[
\frac{d}{dt} X(t) = \hat{V}(t), \quad \frac{d}{dt} V(t) = E(t, X(t)) + \hat{V}(t) \times B(t, X(t)).
\]

These characteristics are combined with suitable initial conditions.

The global existence of solutions given sufficiently regular finite energy initial data remains an open problem. A key result of Glassey and Strauss [7] shows that the solution remains \( C^1 \) as long as the momentum support of \( f \) remains bounded:

**Theorem 1.1** (Glassey and Strauss [7]). Consider initial data \((f_0(x, p), E_0(x), B_0(x))\) which satisfies the constraints (3) such that \( f_0 \in H^5(\mathbb{R}_x^3 \times \mathbb{R}_p^3) \) with compact support in \((x, p)\), \( E_0, B_0 \in H^5(\mathbb{R}_x^3) \) and such that the initial particle density is non-negative, i.e., \( f_0 \geq 0 \). Let \((f, E, B)\) be the unique classical solution to (1)–(3) in \([0, T)\). Assume that there exists a bounded continuous function \( \kappa(t) : [0, T) \to \mathbb{R}_+ \) such that
\[ f(t, x, p) = 0 \text{ for } |p| \geq \kappa(t), \quad \forall x \in \mathbb{R}^3. \]

Then, there exists \( \epsilon > 0 \) such that the solution extends uniquely in \( C^1 \) beyond \( T \) to an interval \([0, T + \epsilon)\).

**Remark 1.1.** We note that as a consequence of the assumptions of the above theorem, the initial energy is bounded
\[
\frac{1}{2} \int_{\mathbb{R}^3} (|E_0|^2 + |B_0|^2) dx + 4\pi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} p_0 f dp dx < \infty,
\]
the initial particle density satisfies
\[
||f_0||_{L_{x, p}^\infty} < \infty,
\]
and the momentum support is initially bounded as
\[
\sup\{|p| : \text{there exists } x \in \mathbb{R}^3 \text{ such that } f_0(x, p) \neq 0 \} < \infty.
\]

We record the above bounds for the initial data explicitly, as they will be useful for the argument of our main theorem below.

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1 The original work [7] actually requires that (6) holds for all approximations of \( f \) instead of only \( f \) itself, and uses initial data with regularity \( f_0 \in C^1(\mathbb{R}_x^3 \times \mathbb{R}_p^3), E_0, B_0 \in C^2(\mathbb{R}_x^3) \). The assumption that (6) holds for all approximations of \( f \) can be removed by a standard application of energy estimates (see, for example, [12]) but this requires slightly more regularity for the initial data (as in the statement of Theorem 1.1).