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The E_8 Moduli 3-Stack of the C-Field in M-Theory

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Abstract: The higher gauge field in 11-dimensional supergravity—the C-field—is constrained by quantum effects to be a cocycle in some twisted version of differential cohomology. We argue that it should indeed be a cocycle in a certain twisted nonabelian differential cohomology. We give a simple and natural characterization of the full smooth moduli 3-stack of configurations of the C-field, the gravitational field/background, and the (auxiliary) E_8-field. We show that the truncation of this moduli 3-stack to a bare 1-groupoid of field configurations reproduces the differential integral Wu structures that Hopkins–Singer had shown to formalize Witten's argument on the nature of the C-field. We give a similarly simple and natural characterization of the moduli 2-stack of boundary C-field configurations and show that it is equivalent to the moduli 2-stack of anomaly free heterotic supergravity field configurations. Finally, we show how to naturally encode the Hořava–Witten boundary condition on the level of moduli 3-stacks, and refine it from a condition on 3-forms to a condition on the corresponding full differential cocycles.

Contents

1. Introduction

The higher gauge fields appearing in string theory (such as the B-field and the RR-fields) and in 11-dimensional M-theory (the C-field) have local presentations by higher degree differential forms that generalize the "vector potential" 1-form familiar from ordinary electromagnetism. However, just as Dirac charge quantization asserts that *globally* the field of electromagnetism is of a more subtle nature, namely given by a connection on a circle bundle, the higher gauge fields in string theory are globally of a more subtle nature: they are cocycles in *differential cohomology* (see for instance [Fr]). Moreover, even this refined statement is strictly true only when each of these fields is considered in isolation. In the full theory they all interact with each other and "twist" or "shift" each other. As a result, generally, the higher gauge fields of string theory are modeled by cocycles in some notion of *twisted differential cohomology*. See [HS,Fr,Sch] for the mathematical background and [DFM,BM,SSS09b,FScSt,FSaSc] for applications in this context. In this article we discuss the differential cohomology of the C-field in 11-dimensional supergravity, twisted by the field of gravity in the bulk of spacetime, as well as by the E_8-gauge field on Hořava–Witten boundaries [HW] and on M5-branes.

The general theory of *twisted differential cohomology* and its characterization of higher gauge fields in string theory is to date only partially understood. For instance, it has been well established that the underlying bare cohomology that controls the interaction of the B-field in type II string theory with the Chan–Paton gauge bundles on D-branes is *twisted K-theory*, and that for trivial B-field the corresponding differential cohomology theory is *differential K-theory*, but a mathematical construction of fully fledged *twisted differential K-theory* has not appeared yet in the literature (see, however, [CMW,KV]). Similarly, partial results apply to the lift of this configuration from type II to M-theory. It is clear that the C-field *in isolation* is modeled by cocycles in degree-4 ordinary differential cohomology, just as the B-field in isolation is modeled by degree-3 differential cohomology, and the electromagnetic field by degree-2 differential cohomology. Less is known about the interaction of the C-field with the degrees of freedom on branes, which here are M5-branes. In our companion article [FSaSc] we investigated aspects of this interaction. The present article provides a detailed discussion of the mathematical model of the C-field, as used there.

The C-field experiences a subtle twist already by its interaction with the field of gravity, via the Spin-structure on spacetime. This was first argued in [Wi97] (we review the argument in Sect. 3.2): *the degree-4 integral class* $[2G]$ *of the C-field is constrained to equal the first fractional Pontrjagin class of the Spin structure modulo the addition of an integral class divisible by* 2. The interpretation of division by 2 in the flux quantization is given in [Sa10b] and related to Wu structures in [Sa11a,Sa11c]. The flux quantization condition can be viewed as defining a twisted String structure [SSS09b]. Dependence of the partition function in M-theory on the Spin structure is investigated in [Sa12a]. Anomalies of M-theory and string theory on manifolds with String structures via E_8 gauge theory is discussed in [Sa11b], and the relation to gerbes is discussed in [Sa10a]. The $\mathbb{Z}_2$-twist of the C-field for a fixed background Spin structure has been formalized in [HS], following an argument in [Wi96,Wi97], by a kind of twisted abelian differential cohomology (which we review in Sect. 3.3). However, two questions remain:

1. On Hořava–Witten boundaries as well as on M5-branes, the C-field interacts with *nonabelian* and in fact *higher* nonabelian gauge fields. *What is the proper refinement of the corresponding twisted differential cohomology* to *non-abelian* differential cohomology?