The Parisi Formula has a Unique Minimizer

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Abstract: In 1979, Parisi (Phys Rev Lett 43:1754–1756, 1979) predicted a variational formula for the thermodynamic limit of the free energy in the Sherrington–Kirkpatrick model, and described the role played by its minimizer. This formula was verified in the seminal work of Talagrand (Ann Math 163(1):221–263, 2006) and later generalized to the mixed $p$-spin models by Panchenko (Ann Probab 42(3):946–958, 2014). In this paper, we prove that the minimizer in Parisi’s formula is unique at any temperature and external field by establishing the strict convexity of the Parisi functional.

1. Introduction and Main Results

The Sherrington–Kirkpatrick (SK) model was introduced in [16]. For any $N \geq 1$, its Hamiltonian at (inverse) temperature $\beta > 0$ and external field $h \in \mathbb{R}$ is given by

$$\frac{\beta}{\sqrt{N}} \sum_{i,j=1}^{N} g_{ij} \sigma_i \sigma_j + h \sum_{i=1}^{N} \sigma_i$$

for $\sigma = (\sigma_1, \ldots, \sigma_N) \in \Sigma_N := \{-1, +1\}^N$, where $g_{ij}$’s are independent standard Gaussian random variables. It is arguably the most well-known model of disordered mean field spin glasses. Over the past few decades, its study has generated hundreds of papers in both theoretical physics and mathematics communities. We refer readers to the book of Mézard et al. [9] for physics’ methodologies and predictions and the books of Talagrand [17] and Panchenko [13] for its recent rigorous treatments.

This paper is concerned with a generalization of the SK model, the so-called mixed $p$-spin model, which corresponds to the Hamiltonian

$$H_N(\sigma) = H'_N(\sigma) + h \sum_{i=1}^{N} \sigma_i$$

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for $\sigma \in \Sigma_N$, where

$$H'_N(\sigma) = \sum_{p=2}^{\infty} \beta_p H_{N,p}(\sigma)$$

is the linear combination of the pure $p$-spin Hamiltonian,

$$H_{N,p}(\sigma) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} g_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p}. \tag{2}$$

Here $g_{i_1,\ldots,i_p}$’s are independent standard Gaussian random variables for all $p \geq 2$ and all $(i_1,\ldots,i_p)$. The nonnegative real sequence $(\beta_p)_{p \geq 2}$ is called the temperature parameters and $h \in \mathbb{R}$ denotes the strength of the external field. We assume that $\beta_p > 0$ for at least one $p \geq 2$ and $(\beta_p)_{p \geq 2}$ decreases fast enough, for instance, $\sum_{p=2}^{\infty} 2^p \beta_p^2 < \infty$. The SK model can be recovered by choosing $\beta_p = 0$ for all $p \geq 3$. A direct computation gives

$$\mathbb{E} H'_N(\sigma^1) H'_N(\sigma^2) = N \xi(R_{1,2}),$$

where $R_{1,2} := N^{-1} \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2$ is the overlap between spin configurations $\sigma^1$ and $\sigma^2$ and

$$\xi(s) := \sum_{p=2}^{\infty} \beta_p^2 s^p, \quad \forall s \in [-1, 1]. \tag{3}$$

Define the Gibbs measure as $G_N(\sigma) = Z_N^{-1} \exp(-H_N(\sigma))$ for $\sigma \in \Sigma_N$, where the normalizing factor $Z_N$ is known as the partition function.

Let $\mathcal{M}$ be the collection of all probability measures on $[0, 1]$ endowed with the metric $d(\mu, \mu') := \int_0^1 |\mu([0, s]) - \mu'([0, s])| ds$. Denote by $\mathcal{M}_d$ the collection of all atomic measures from $\mathcal{M}$. Let $\mu \in \mathcal{M}_d$ with jumps at $\{q_p\}_{p=1}^{k+1}$ for some $k \geq 0$. Set $q_0 = 0$, $q_{k+2} = 1$ and $m_l = \mu([q_l, q_{l+1}])$ for $0 \leq l \leq k + 1$. We set a real-valued function $\Phi_\mu$ on $[0, 1] \times \mathbb{R}$ as follows. Starting with $\Phi_\mu(1, x) = \Phi_\mu(q_{k+2}, x) = \log \cosh x$, define for $(s, x) \in [q_{k+1}, q_{k+2}) \times \mathbb{R},$

$$\Phi_\mu(s, x) = \frac{1}{m_{k+1}} \log \mathbb{E} \exp m_{k+1} \Phi_\mu(q_{k+2}, x + z \sqrt{\xi(q_{k+2}) - \xi'(s)})$$

$$= \log \cosh x + \frac{1}{2} (\xi'(1) - \xi'(s)) \tag{4}$$

and decreasingly for $0 \leq l \leq k$,

$$\Phi_\mu(s, x) = \frac{1}{m_l} \log \mathbb{E} \exp m_l \Phi_\mu(q_{l+1}, x + z \sqrt{\xi(q_{l+1}) - \xi'(s)}) \tag{5}$$

for $(s, x) \in [q_l, q_{l+1}) \times \mathbb{R}$, where $z$ is a standard Gaussian random variable. Two important properties about $\Phi_\mu$ are the following. First, it satisfies the Parisi PDE on each $[q_l, q_{l+1}) \times \mathbb{R},$

$$\partial_s \Phi_\mu(s, x) = -\frac{\xi''(s)}{2} \left( \partial_{xx} \Phi_\mu(s, x) + \mu([0, s]) (\partial_x \Phi_\mu(s, x))^2 \right). \tag{6}$$