Application of the Lace Expansion to the $\phi^4$ Model

Akira Sakai

Department of Mathematics, Hokkaido University, Sapporo, Japan.
E-mail: sakai@math.sci.hokudai.ac.jp

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Abstract: Using the Griffiths–Simon construction of the $\phi^4$ model and the lace expansion for the Ising model, we prove that, if the strength $\lambda \geq 0$ of nonlinearity is sufficiently small for a large class of short-range models in dimensions $d > 4$, then the critical $\phi^4$ two-point function $\langle \phi_0 \phi_x \rangle_{\mu_c}$ is asymptotically $|x|^{2-d}$ times a model-dependent constant, and the critical point is estimated as $\mu_c = \hat{J} - \frac{\lambda}{2} \langle \phi^2_0 \rangle_{\mu_c} + O(\lambda^2)$, where $\hat{J}$ is the massless point for the Gaussian model.

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1. Introduction and the Main Results

The (lattice) $\phi^4$ model is a pedagogical yet nontrivial model in scalar field theory. It is also considered to be an interface model defined by a Hamiltonian having a quartic self-energy term. (See, e.g., [7] for recent developments in another class of interface models,
(called gradient fields.) If that quartic term is absent, then it becomes the Gaussian model and its two-point function satisfies the same convolution equation as the random-walk’s Green function. In particular, for the massless Gaussian model, which is a lattice version of Gaussian free fields, the two-point function decays as a multiple of $|x|^{2-d}$ as $|x| \to \infty$ when $d > 2$.

On the other hand, the $\varphi^4$ two-point function is known to satisfy a nonlinear equation, called the Schwinger–Dyson equation. The nonlinearity is due to involvement of four-spin expectations. This implies that, in order to find the exact expression for the two-point function, we must also know the exact expressions for four-spin expectations. In general, the Schwinger–Dyson equation for $2n$-spin expectations involves $(2n+2)$-spin expectations. Therefore, it is seemingly impossible to solve those infinitely many simultaneous equations to find the exact expression for the two-point function.

Instead of solving those simultaneous equations, there have been many useful ideas to study the phase transition and critical behavior for the $\varphi^4$ model. Among those are the use of reflection positivity [11, 12] and correlation inequalities obtained by the random-current representation [3, 4]. They imply that, for the nearest-neighbor model in dimensions $d > 2$, there is a nontrivial critical point $\mu_c \in \mathbb{R}$ such that the two-point function $\langle \varphi_0 \varphi_x \rangle_{\mu}$ is bounded above by a multiple of $|x|^{2-d}$ uniformly in $\mu > \mu_c$, and therefore all critical exponents take on their mean-field values in dimensions $d > 4$ [1, 10, 29] (see also [9] and references therein). Moreover, for the nearest-neighbor model, the rigorous renormalization-group (RG) approach based on the block-spin transformation [13, 14, 16] may identify an asymptotic expression for the critical two-point function $\langle \varphi_0 \varphi_x \rangle_{\mu_c}$, which is presumably $C|x|^{2-d}$ as $|x| \to \infty$ for some constant $C \in (0, \infty)$. This is proven to be affirmative when $d = 4$ (cf., [8, Theorem I.2] and [15, (8.32)]; see also [2] for the recent RG results on the $n$-component $|\varphi|^4$ model in 4 dimensions). However, as far as we are aware, such strong results have not been reported in dimensions $d > 4$.

For the Ising model, which is considered to be in the same universality class as the $\varphi^4$ model, we have been able to show [25] that, not only for the nearest-neighbor model but also for a large class of spread-out models which do not necessarily satisfy reflection positivity, the critical Ising two-point function is asymptotically a model-dependent multiple of $|x|^{2-d}$, if the dimension $d$ or the range of spin-spin coupling is sufficiently large. The proof is based on the lace expansion, which was first applied to weakly self-avoiding walk [5] and then developed for lattice trees and lattice animals [19], percolation [20], oriented percolation [23] and the contact process [24]. The asymptotic behavior of the critical two-point function for each spacial model is proved in [17, 18, 25]. The methodology has been extended to cover the case of power-law decaying spin-spin coupling [6] (see also [21] for results in the Fourier space).

In this paper, we apply the lace expansion for the Ising model to prove asymptotic behavior of the $\varphi^4$ two-point function. In order to do so, we first use the Griffiths–Simon construction [27] to approximate each $\varphi^4$ spin by a sum of $N$ Ising-spin variables. This is a well-known approach to study the $\varphi^4$ model (see, e.g., [1, Section 10]). Then, we investigate the lace-expansion coefficients and determine the right scaling in powers of $N$. As a result, we prove the expected asymptotic behavior of the critical two-point function, i.e., $\langle \varphi_0 \varphi_x \rangle_{\mu_c} \sim \exists C|x|^{2-d}$ as $|x| \to \infty$, for a large class of short-range models on $\mathbb{Z}^d$ with $d > 4$, if the strength $\lambda \geq 0$ of nonlinearity is sufficiently small. This implies triviality of the continuum limit, as pointed out in [10, Section 7] (see also [1]). During the course, we also obtain the $\lambda$-expansion of the critical point $\mu_c$ up to $O(\lambda^2)$ around the massless point for the Gaussian model.