Onsager’s Conjecture Almost Everywhere in Time

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Abstract: In recent works by Isett (Hölder continuous Euler flows in three dimensions with compact support in time, pp 1–173, 2012), and later by Buckmaster et al. (Ann Math 2015), iterative schemes were presented for constructing solutions belonging to the Hölder class $C^{1/5-\varepsilon}$ of the 3D incompressible Euler equations which do not conserve the total kinetic energy. The cited work is partially motivated by a conjecture of Lars Onsager in 1949 relating to the existence of $C^{1/3-\varepsilon}$ solutions to the Euler equations which dissipate energy. In this note we show how the later scheme can be adapted in order to prove the existence of non-trivial Hölder continuous solutions which for almost every time belong to the critical Onsager Hölder regularity $C^{1/3-\varepsilon}$ and have compact temporal support.

0. Introduction

In what follows $\mathbb{T}^3$ denotes the 3-dimensional torus, i.e. $\mathbb{T}^3 = S^1 \times S^1 \times S^1$. Formally, we say $(v, p)$ solves the incompressible Euler equations if

$$\begin{cases} \partial_t v + \text{div} v \otimes v + \nabla p = 0 \\ \text{div} v = 0 \end{cases}. \quad (1)$$

Suppose $v$ is such a solution, then we define its kinetic energy, as

$$E(t) := \frac{1}{2} \int_{\mathbb{T}^3} |v(x, t)|^2 \, dx.$$

A simple calculation applying integration by parts yields that for any classical solution of (1) the kinetic energy is in fact conserved in time. This formal calculation does not however hold for distributional solutions to Euler (cf. [6,7,9,20–22]).

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In fact in the context of 3-dimensional turbulence, flows *dissipating* energy in time have long been considered. A key postulate of Kolmogorov’s K41 theory [17] is that for homogeneous, isotropic turbulence, the dissipation rate is non-vanishing in the inviscid limit. In particular, defining the *structure functions* for homogeneous, isotropic turbulence

\[ S_p(\ell) := \left\langle \left( (v(x + \hat{\ell}) - v(x)) \cdot \frac{\hat{\ell}}{\ell} \right)^p \right\rangle, \]

where \( \hat{\ell} \) denotes a spatial vector of length \( \ell \), Kolmogorov’s famous four-fifths law can be stated as

\[ S_3(\ell) = -\frac{4}{5} \varepsilon_d \ell, \tag{2} \]

where here \( \varepsilon_d \) denotes the mean energy dissipation per unit mass. More generally, Kolmogorov’s scaling laws can be stated as

\[ S_p(\ell) = C_p \varepsilon_d^{p/3} \ell^{p/3}, \tag{3} \]

for any positive integer \( p \).

A well known consequence of the above scaling laws is the Kolmogorov spectrum, which postulates a scaling relation on the ‘energy spectrum’ of a turbulent flow (cf. [13,15]). It was this observation that provided motivation for Onsager to conjecture in his famous note [19] on statistical hydrodynamics, the following dichotomy:

(a) Any weak solution \( v \) belonging to the Hölder space \( C^\theta \) for \( \theta > \frac{1}{3} \) conserves the energy.

(b) For any \( \theta < \frac{1}{3} \) there exist weak solutions \( v \in C^\theta \) which do not conserve the energy.

Part (a) of this conjecture has since been resolved: it was first considered by Eyink in [12] following Onsager’s original calculations and later proven by Constantin, E and Titi in [5]. Subsequently, this later result was strengthened by showing that under weakened assumptions on \( v \) (in terms of Besov spaces) kinetic energy is conserved [4,11].

Part (b) remains an open conjecture and is the subject of this note. The first constructions of non-conservative Hölder-continuous \( (C^{1/10 - \varepsilon}) \) weak solutions appeared in the work of Lellis and Székelyhidi [8], which itself was based on their earlier seminal work [10] where continuous weak solutions were constructed. Furthermore, it was shown in the mentioned work that such solutions can be constructed obeying any prescribed smooth non-vanishing energy profile. In recent work [16], Isett introduced a number of new ideas in order to construct non-trivial \( 1/5 - \varepsilon \) Hölder-continuous weak solutions with compact temporal support. This construction was later improved by Buckmaster, et al. [2], following more closely the earlier work [8,10], in order construct \( 1/5 - \varepsilon \) Hölder-continuous weak solution obeying a given energy profile.

In this note we give a proof of the following theorem.

**Theorem 0.1.** There exists a non-trivial continuous vector field \( v \in C^{1/5 - \varepsilon} (\mathbb{T}^3 \times (-1, 1), \mathbb{R}^3) \) with compact support in time and a continuous scalar field \( p \in C^{2/5 - 2\varepsilon} (\mathbb{T}^3 \times (-1, 1)) \) with the following properties:

(i) The pair \( (v, p) \) solves the incompressible Euler equations (1) in the sense of distributions.