A Mathematical Theory of Super-Resolution by Using a System of Sub-Wavelength Helmholtz Resonators

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Received: 2 June 2014 / Accepted: 12 November 2014
Published online: 14 February 2015 – © Springer-Verlag Berlin Heidelberg 2015

Abstract: A rigorous mathematical theory is developed to explain the super-resolution phenomenon observed in the experiment (Lemoult et al., Phys Rev Lett 107:064301, 2011). A key ingredient is the calculation of the resonances and the Green function in the half space with the presence of a system of Helmholtz resonators in the quasi-stationary regime. By using boundary integral equations and generalized Rouché’s theorem, the existence and the leading asymptotic of the resonances are rigorously derived. The integral equation formulation also yields the leading order terms in the asymptotics of the Green function. The methodology developed in the paper provides an elegant and systematic way for calculating resonant frequencies for Helmholtz resonators in assorted space settings, as well as in various frequency regimes. By using the asymptotics of the Green function, the analysis of the imaging functional of the time-reversal wave fields becomes possible, which clearly demonstrates the super-resolution property. The result provides the first mathematical theory of super-resolution in the context of a deterministic medium and sheds light on the mechanism of super-resolution and super-focusing for waves in deterministic complex media.

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This work was supported by the ERC Advanced Grant Project MULTIMOD-267184.
1. Introduction

When light is focused by the objective of a microscope, the notion of light rays converging to an infinitely sharp focal point does not hold true. Instead, as observed by Abbe [1], the light wave forms a blurred or diffracted focal spot with a finite size due to diffraction. The size of the spot depends on the wavelength of the light and the angle at which the light wave converges; the latter is, in turn, determined by the numerical aperture of the objective. Similarly, a point emitter also appears as a blurred spot, and the size of the spot places a fundamental limit on the minimal distance at which we can resolve two emitters. The intensity profile of this spot, which defines the point spread function of the microscope, has approximately the same width as that of the focal spot described above. Consequently, two identical emitters separated by a distance less than the width of the point spread function will appear as a single object, making them unresolvable from each other [2,29]. This resolution limit, referred to as the Abbe–Rayleigh or the diffraction limit of resolution, applies only to light that has propagated for a distance substantially larger than its wavelength [8,9]. It is well-known since the seminal work of Synge [40] that near-field microscopes achieve resolutions well below the diffraction limit.

Discerning features that are spectrally disparate is not challenging by diffraction. It is now well-established that spectroscopic imaging can yield super-resolution [20]. Likewise, Abbe’s barrier does not prevent finding out the location of a point emitter with arbitrary precision [16,17,28]. Breaking Abbe’s barrier is only about discerning features within a distance smaller than Abbe’s barrier.

Since the mid-20th century, several approaches aimed at pushing the diffraction limits by reducing the focal spot size. In the optical domain, the sub-wavelength-scaled resonant media capable of beating the classical diffusion limit and the concepts such as superlenses [18], imaging single molecules [28], and super-oscillations [13], could provide feasible alternatives [42].